Workshop on Symplectic Field Theory IX: POLYFOLDS FOR SFT

Lectures 5 - 9 (version 1.0)

University of Augsburg

27 - 31 August 2018

Preamble:

Please direct corrections, comments, etc. to

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Work in progress:

`www.polyfolds.org`

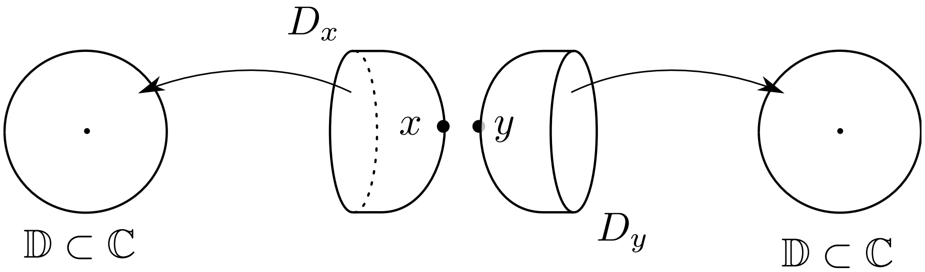
Hopeful idea: Polyfold Summer School

Topics:

1. Toy-model M-polyfold (standard node)
2. Imprinting method (theory & example)
3. Imprinting plus operations
4. A basic "LEGO" block
5. New blocks from old (theory & example)
6. Periodic orbits and nodal interface pairs
7. Preliminary "LEGO" building

Nodal Disk Pair

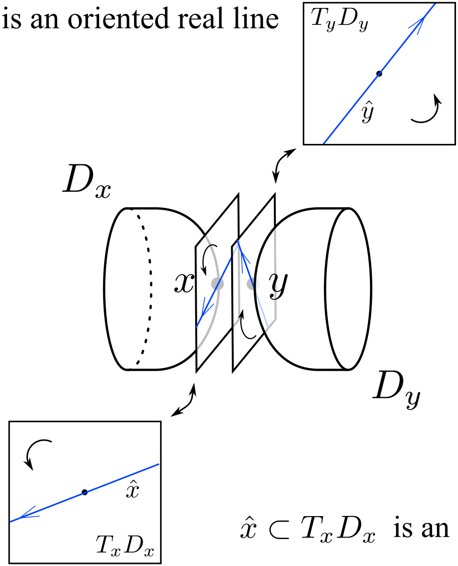
$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$



Nodal Disk Pair

$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$

$\hat{y} \subset T_y D_y$ is an oriented real line



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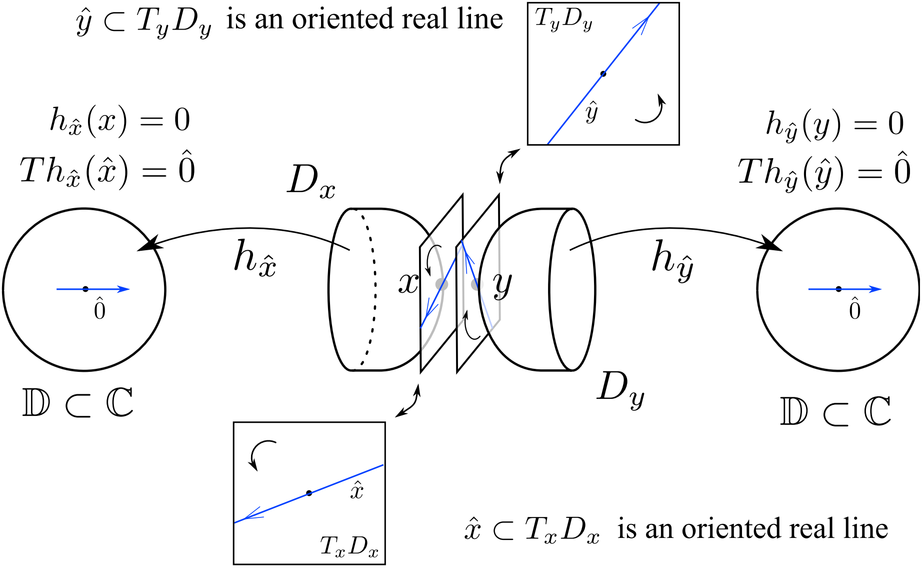


$\hat{x} \subset T_x D_x$ is an oriented real line

Natural angles

Circle action on decorations:

$$(\theta, \hat{x}) \to \theta * \hat{x} := e^{2\pi i\theta} \hat{x}$$

Equivalence relation on decorated nodal pairs:

$$\{\hat{x}, \hat{y}\} \sim \{\hat{x}', \hat{y}'\} \text{ iff } \exists \quad \theta \in S^1 = \mathbb{R}/\mathbb{Z}$$

such that

$$\hat{x}' = \theta * \hat{x} \text{ and } \hat{y}' = \theta^{-1} * \hat{y}$$

Natural angles

Circle action on decorations:

$$(\theta, \hat{x}) \to \theta * \hat{x} := e^{2\pi i\theta}\hat{x}$$

A natural angle is then defined as
an element in the associated equivalence class,
or alternatively as

$$[\hat{x}, \hat{y}] = \left\{ \{\theta * \hat{x}, \theta^{-1} * \hat{y}\} : \theta \in S^1 \right\}$$

Gluing Paremeters

Associated to a nodal disk pair

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

we define the associated set of gluing parameters

$$\mathbb{B}_{\mathcal{D}}$$

as formal expressions of the form

$$r \cdot [\hat{x}, \hat{y}]$$

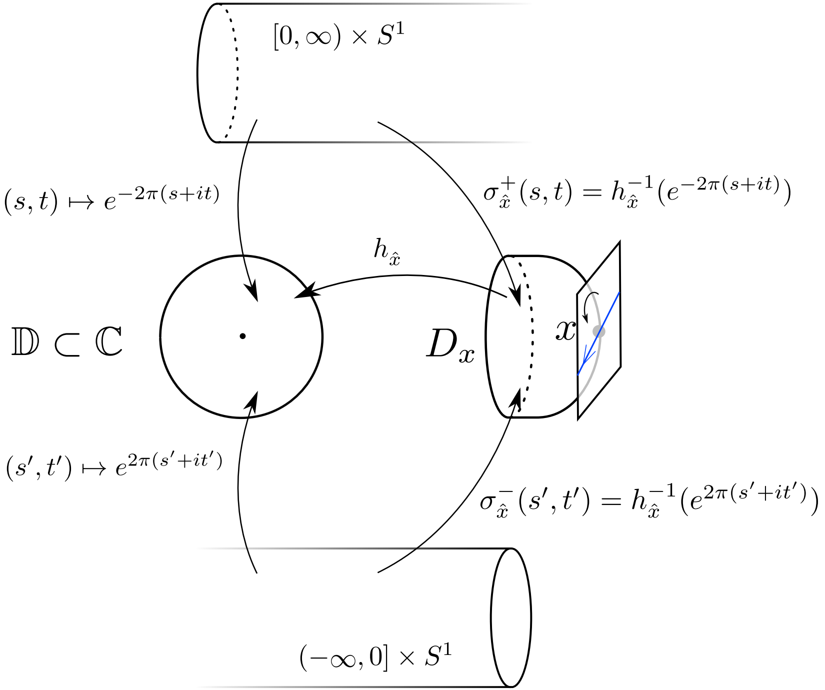
Cylinders Z_a

Given a nodal disk pair $\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$ and a gluing parameter $a = r \cdot [\hat{x}, \hat{y}] \in \mathbb{B}_{\mathcal{D}}$ with $r > 0$ define the cylinder

$$Z_a = \Big\{\{z, z'\} : z \in D_x,\ z' \in D_y,\\ h_{\hat{x}}(z) \cdot h_{\hat{y}}(z') = e^{-2\pi\varphi(r)}\Big\}$$

for $a = 0$ i.e. $r = 0$ define

$$Z_a = D_x \sqcup D_y$$


$[0,\infty)\times S^1$
$(s,t)\mapsto e^{-2\pi(s+it)}$
$\sigma^+_{\hat{x}}(s,t)=h^{-1}_{\hat{x}}(e^{-2\pi(s+it)})$
$h_{\hat{x}}$
$\mathbb{D}\subset\mathbb{C}$
D_x
x
$(s',t')\mapsto e^{2\pi(s'+it')}$
$\sigma^-_{\hat{x}}(s',t')=h^{-1}_{\hat{x}}(e^{2\pi(s'+it')})$
$(-\infty,0]\times S^1$

Cylinders Z_a

The maps

$$\sigma_{\hat{x}}^{+} : [0, \infty) \times S^1 \to D_x$$
$$\sigma_{\hat{y}}^{-} : (-\infty, 0] \times S^1 \to D_y$$

induces coordinates on D_x and D_y via

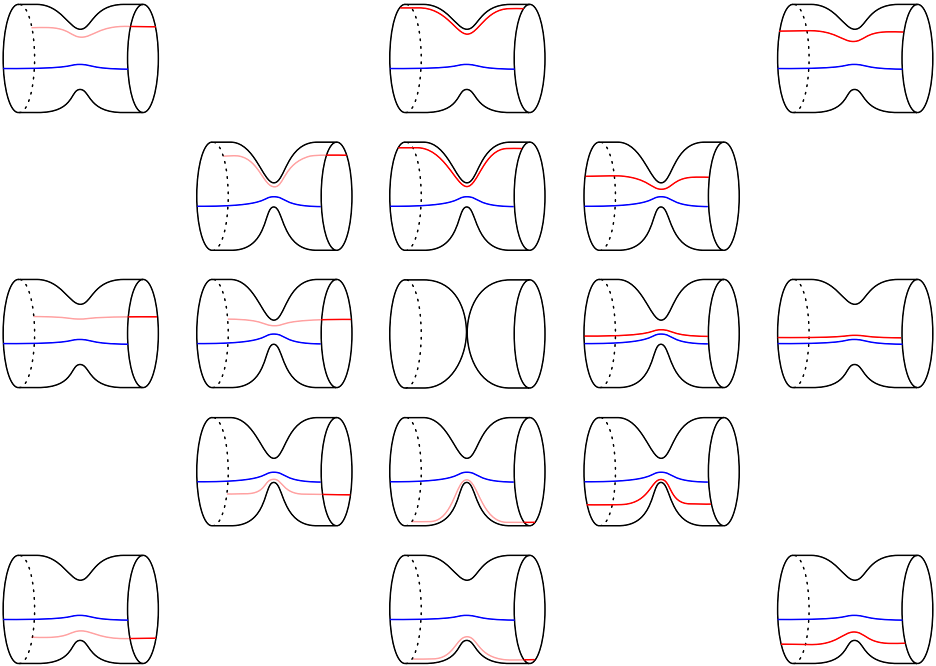
$$z = (s, t) \in [0, \infty) \times S^1 \quad \text{for} \quad z \in D_x$$
$$z' = (s', t') \in (-\infty, 0] \times S^1 \quad \text{for} \quad z' \in D_y$$

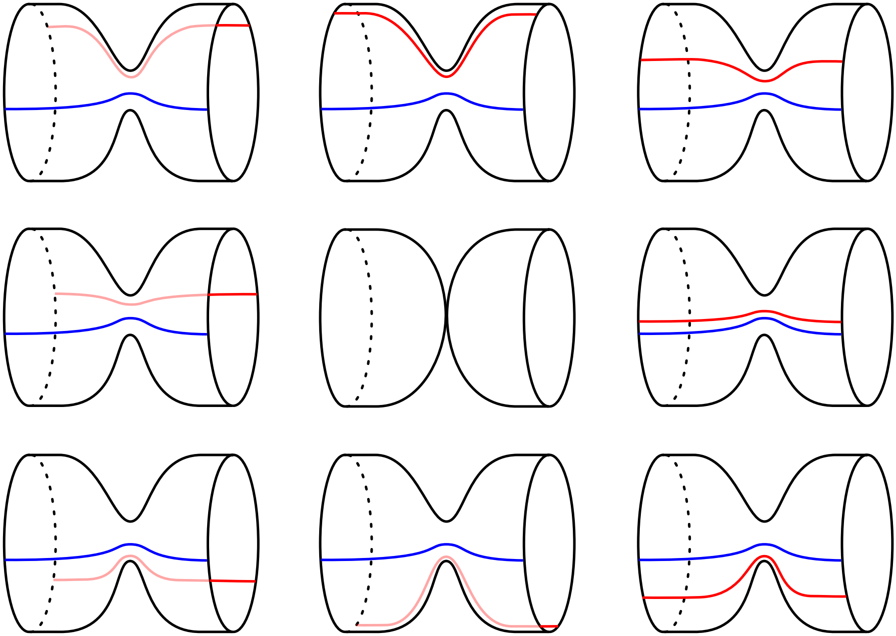
Cylinders Z_a

These induce coordinates on the Z_a which can alternately be described as

$$\begin{aligned} Z_a = \big\{\{(s,t),(s',t')\} : (s,t) &\in [0,R] \times S^1, \\ (s',t') &\in [-R,0] \times S^1 \\ s &= s' + R, \\ t &= t' + \theta\big\} \end{aligned}$$

where $R = \varphi(|a|)$





Cylinders Z_a Takeaway

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\}) \rightsquigarrow \mathbb{B}_{\mathcal{D}}$$

$$\left(\mathbb{B}_{\mathcal{D}}, \mathcal{D}\right) \rightsquigarrow \bigcup_{a \in \mathbb{B}_{\mathcal{D}}} Z_a$$

$$a \neq a' \implies Z_a \neq Z_{a'}$$

$$a = r \cdot [\hat{x}, \hat{y}] \in \mathbb{B}_{\mathcal{D}}$$

Disconnected Function Spaces

$$\delta : 0 < \delta_0 < \delta_1 < \cdots$$

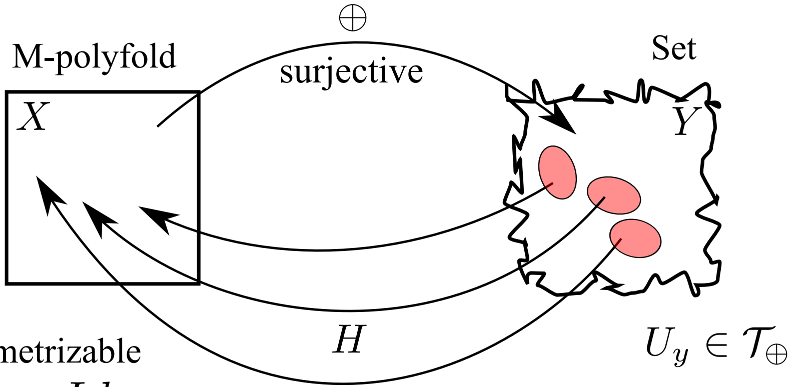
$$E_{\mathcal{D}}^{\delta_0} = \mathbb{R}^N \oplus H^{3,\delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

$$X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) = E_{\mathcal{D}}^{\delta_0} \sqcup \Big(\bigcup_{0<|a|<\frac{1}{4}} H^3(Z_a, \mathbb{R}^N)\Big)$$

We aim to equip $X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)$ with an M-polyfold structure

Theorem: Imprinting Method

Given:



$\mathcal{T}_\oplus$ is metrizable

$\oplus \circ H = Id$

$H \circ \oplus$ is sc$^\infty$

Then:

- Y is an M-polyfold
- $\oplus$ and each H is sc$^\infty$

Specific Imprinting

$$\oplus : \mathbb{B}_{\mathcal{D}} \times E^{\delta_0}_{\mathcal{D}} \to X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N)$$

$$\oplus_a(u^+, u^-) : Z_a \to \mathbb{R}^N$$

$$\oplus_a(u^+, u^-)(\{(s,t),(s',t')\}) =$$

$$\beta(|s| - \tfrac{1}{2}R) \cdot u^+(s,t) + \beta(|s'| - \tfrac{1}{2}R) \cdot u^-(s',t')$$

Recall:

$$E^{\delta_0}_{\mathcal{D}} = \mathbb{R}^N \oplus H^{3,\delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

$$X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N) = E^{\delta_0}_{\mathcal{D}} \sqcup \Big(\bigcup_{0<|a|<\frac{1}{4}} H^3(Z_a, \mathbb{R}^N)\Big)$$

Housekeeping Theorem 1

Given:

Imprintings

$$\oplus_1 : X_1 \to Y_1$$
$$\oplus_2 : X_2 \to Y_2$$

Then:

$$\oplus_1 \times \oplus_2 : X_1 \times X_2 \to Y_1 \times Y_2$$
$$\oplus_1 \sqcup \oplus_2 : X_1 \sqcup X_2 \to Y_1 \sqcup Y_2$$

are imprintings

Example: Disjoint Union

Given:

two nodal disk pairs

$$\mathcal{D}_1 = (D_{x_1} \sqcup D_{y_1}, \{x_1, y_1\})$$

$$\mathcal{D}_2 = (D_{x_2} \sqcup D_{y_2}, \{x_2, y_2\})$$

and imprintings

$$\oplus_1 : \mathbb{B}_{\mathcal{D}_1} \times E^{\delta_0}_{\mathcal{D}_1} \to X^{\delta_0}_{\mathcal{D}_1,\varphi}(\mathbb{R}^N)$$

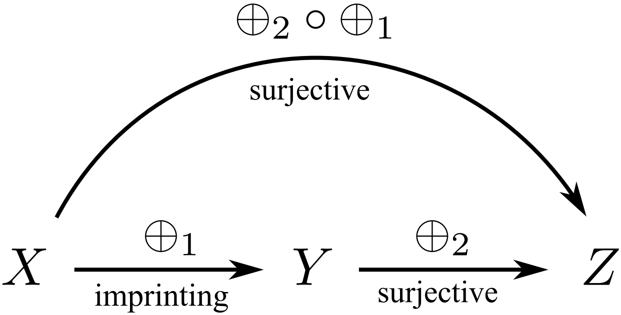
$$\oplus_2 : \mathbb{B}_{\mathcal{D}_2} \times E^{\delta_0}_{\mathcal{D}_2} \to X^{\delta_0}_{\mathcal{D}_2,\varphi}(\mathbb{R}^N)$$

Then:

- $X^{\delta_0}_{\mathcal{D}_1,\varphi}(\mathbb{R}^N) \sqcup X^{\delta_0}_{\mathcal{D}_2,\varphi}(\mathbb{R}^N)$ is an M-polyfold
- $\oplus_1 \bigsqcup \oplus_2$ is an imprinting

Housekeeping Theorem 2

Given:

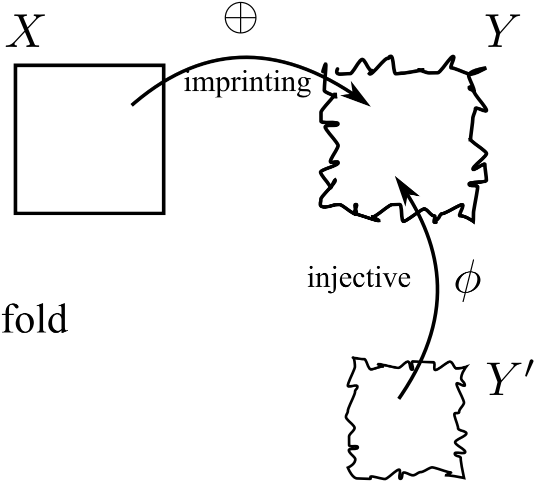


Then:

- $\bigoplus_2$ is an imprinting if and only if
- $\bigoplus_2 \circ \bigoplus_1$ is an imprinting
- Moreover: coherence.

Housekeeping Theorem 3

Given:



$X' := \oplus^{-1}(\phi(Y'))$
is a sub-M-polyfold

Then:

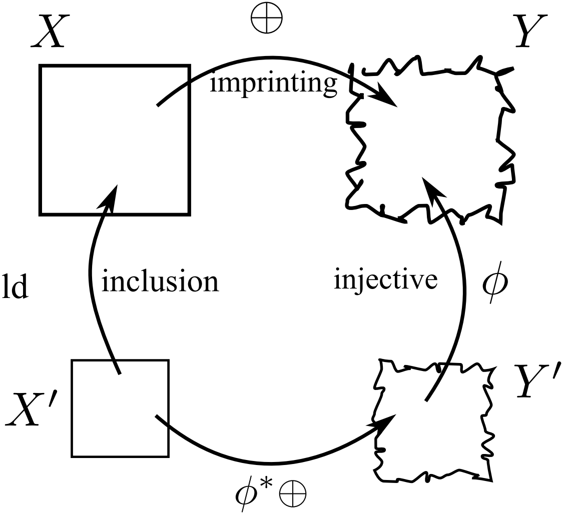
$\phi^*\oplus : X' \to Y'$ is an imprinting, where

$$\phi^*\oplus := \phi^{-1} \circ \oplus\big|_{X'}$$

Housekeeping Theorem 3

Given:

$X' := \oplus^{-1}(\phi(Y'))$
is a sub-M-polyfold

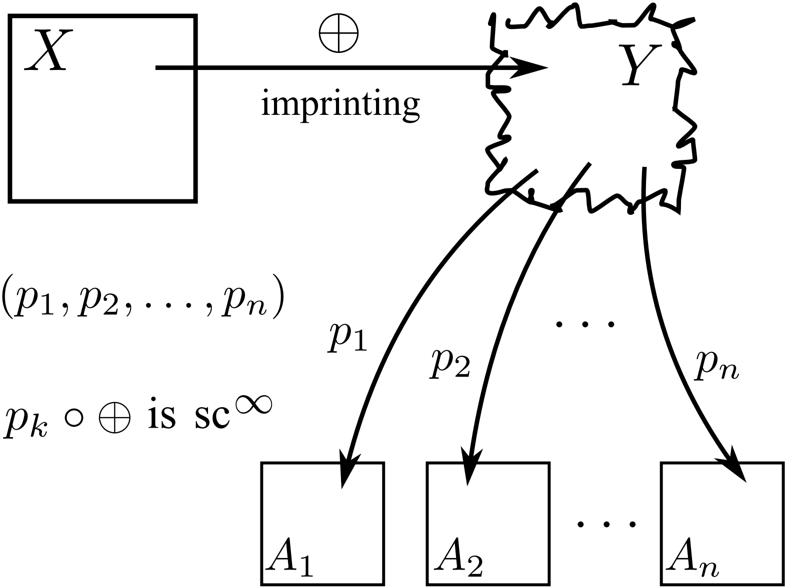


Then:

$\phi^*\oplus : X' \to Y'$ is an imprinting, where

$$\phi^*\oplus := \phi^{-1} \circ \oplus\big|_{X'}$$

Imprinting with restrictions $(\oplus, \mathbf{p})$

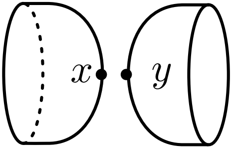


$\mathbf{p} = (p_1, p_2, \dots, p_n)$

each $p_k \circ \oplus$ is sc^∞

Imprinting with restrictions -- Example

Recall, the nodal disk pair



$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$

gives rise to the imprinting

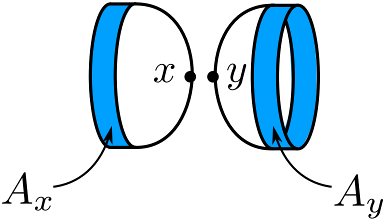
$$\oplus : \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \to X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)$$

$$E_{\mathcal{D}}^{\delta_0} = \mathbb{R}^N \oplus H^{3,\delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

$$X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) = E_{\mathcal{D}}^{\delta_0} \sqcup \Big(\bigcup_{0<|a|<\frac{1}{4}} H^3(Z_a, \mathbb{R}^N)\Big)$$

Imprinting with restrictions -- Example

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

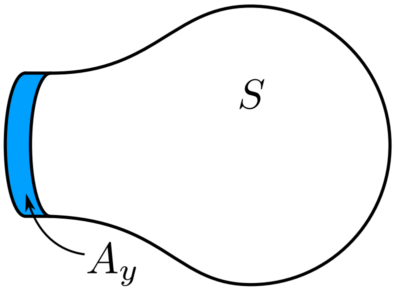


yields and imprinting with restrictions

$$\mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \xrightarrow{\;\bigoplus\;} X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)$$

$$p_x : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_x, \mathbb{R}^N), \qquad p_y : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_y, \mathbb{R}^N)$$

Imprinting with restrictions -- Example



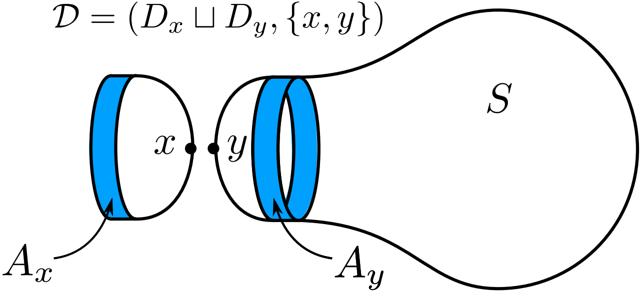
yields and imprinting with restrictions

$$\mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \xrightarrow{\oplus} X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \qquad H^3(S, \mathbb{R}^N)$$

$$p_x : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_x, \mathbb{R}^N), \quad p_y : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_y, \mathbb{R}^N), \quad p'_y : H^3(S, \mathbb{R}^N) \to H^3(A_y, \mathbb{R}^N)$$

Imprinting with restrictions -- Example

$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$

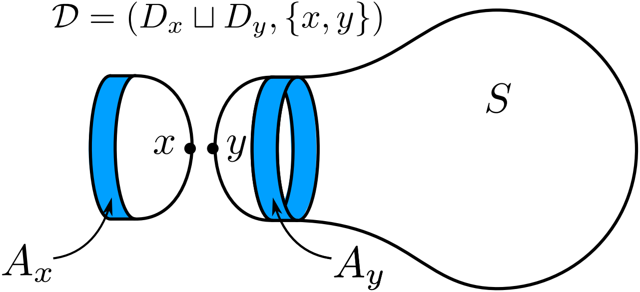


yields the M-polyfold

$$X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N)\ {}_{p_y}\!\times_{p'_y}\ H^3(S, \mathbb{R}^N)$$

Imprinting with restrictions -- Example

$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$



and more importantly yields an imprinting with restrictions

$$(\oplus \times Id)^{-1}\big(X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N)\ {}_{p_y}\!\times_{p'_y}\ H^3(S, \mathbb{R}^N)\big) \xrightarrow{\phi^*(\oplus \times Id)} X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N)\ {}_{p_y}\!\times_{p'_y}\ H^3(S, \mathbb{R}^N) \xrightarrow{p_x} H^3(A_x, \mathbb{R}^N)$$

where ϕ is the inclusion

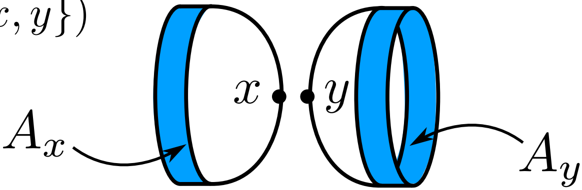
$$X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N)\ {}_{p_y}\!\times_{p'_y}\ H^3(S, \mathbb{R}^N) \xrightarrow{\phi} X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N) \times H^3(S, \mathbb{R}^N)$$

Imprinting with restrictions -- Theorem

The fiber product over annular restrictions of imprintings with restrictions, is again an imprinting with restrictions

Feature: Projection to gluing parameter

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$



$$\begin{array}{ccccc}
\mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} & \xrightarrow{\ \bigoplus\ } & X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) & \xrightarrow{\ p_{\mathbb{B}_{\mathcal{D}}}\ } & \mathbb{B}_{\mathcal{D}}
\end{array}$$

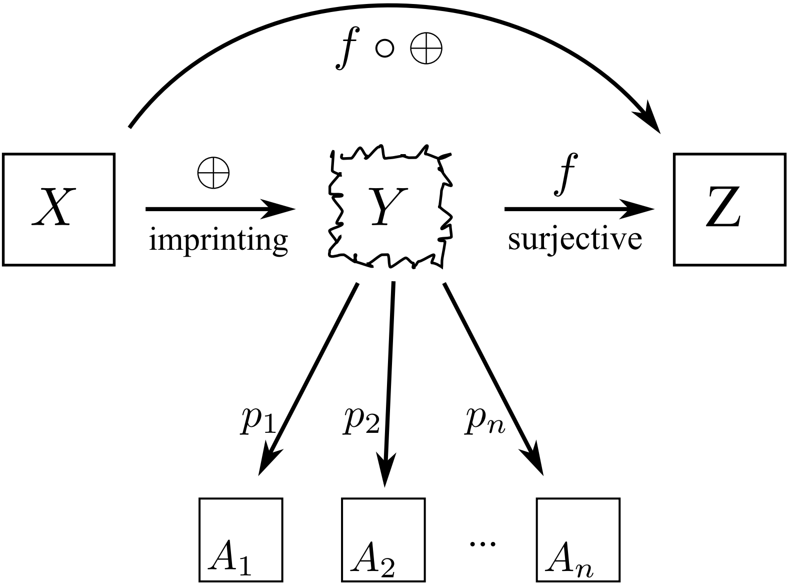
$$p_x : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_x, \mathbb{R}^N), \qquad p_y : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_y, \mathbb{R}^N)$$

Definition: Submersive imprinting w. restrictions

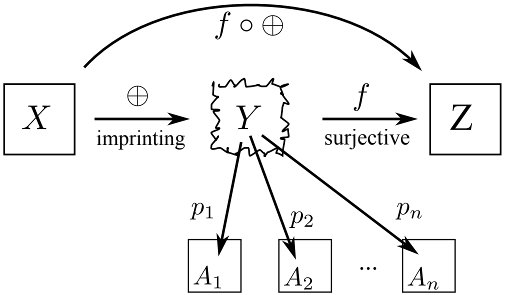
Definition: Submersive imprinting w. restrictions

Basic LEGO block

Definition: Basic LEGO Block



Definition: Basic LEGO Block



For each $\big(x_0, f \circ \oplus(x_0)\big) \in \mathrm{Gr}(f \circ \oplus) \subset X \times Z$

there exists an open nbhd $W \subset X \times Z$ and sc-smooth map

$\rho : W \to W$ of the form $\rho(x,z) = \big(\bar{\rho}(x,z), z\big)$ such that

$$\rho \circ \rho = \rho$$

$$\rho(W) = W \cap \mathrm{Gr}(f \circ \oplus)$$

$$p_i \circ \oplus \circ \bar{\rho}(x,z) = p_i(x)$$

Benefits of LEGO blocks:

Given LEGO blocks $(\oplus, \mathbf{p}, f)$ and $(\oplus', \mathbf{p}', f')$ the fiber product over f and f' is another LEGO block.

If the $\mathbf{p}$ and $\mathbf{p}'$ are restrictions to annular neighborhoods, then the fiber product over elements of the $\mathbf{p}$ and $\mathbf{p}'$ is also another LEGO block.

From $\mathbb{R}^N$ to manifolds

With $X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N)$ defined, we now aim to define $X^{\delta_0}_{\mathcal{D},\varphi}(Q)$ where Q is a manifold.

Let

- $\Phi : Q \to \mathbb{R}^N$ be an embedding
- $U \subset \mathbb{R}^N$ be an open neighborhood of $\Phi(Q)$
- $\mathrm{pr} : U \to U$ a smooth retraction onto $\Phi(Q)$

i.e. $\mathrm{pr} \circ \mathrm{pr} = \mathrm{pr} \qquad \mathrm{pr}(U) = \Phi(Q)$

From $\mathbb{R}^N$ to manifolds

Then $\mathcal{U} := \{u \in X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N) : \mathrm{Im}(u) \subset U\}$
is open and the map

$$\rho : \mathcal{U} \to \mathcal{U}$$
$$\rho(u) = \mathrm{pr} \circ u$$

is an sc-smooth retraction.

This defines an M-polyfold structure on

$$X^{\delta_0}_{\mathcal{D},\varphi}(Q)_{\Phi,\mathbb{R}^N} = \bigcup_{a \in \mathbb{B}_{\mathcal{D}}} \left\{u \in \mathcal{C}^0(Z_a, Q) : \Phi \circ u \in \rho(\mathcal{U})\right\}$$

moreover $X^{\delta_0}_{\mathcal{D},\varphi}(Q)_{\Phi,\mathbb{R}^N} = X^{\delta_0}_{\mathcal{D},\varphi}(Q)_{\Phi',\mathbb{R}^{N'}}$ as M-polyfolds,
so we simply write $X^{\delta_0}_{\mathcal{D},\varphi}(Q)$

The periodic orbit case

Introduce

- periodic orbit: $\boldsymbol{\gamma} = ([\gamma], T, k)$
- weighted periodic orbit $\bar{\boldsymbol{\gamma}} = (\boldsymbol{\gamma}, \delta)$

  with $\delta = (\delta_k)_{k=0}^{\infty}$
- ordered nodal disk pair

  $$\mathcal{D} = \big(D_x \sqcup D_y, (x, y)\big)$$

The periodic orbit case

We define the function space $Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\boldsymbol{\gamma}})$ to be the set of tuples $(\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)$ where

$$\tilde{u}^x : D_x \setminus \{x\} \to \mathbb{R}^N$$
$$\tilde{u}^y : D_y \setminus \{y\} \to \mathbb{R}^N$$

$[\hat{x}, \hat{y}]$ is a natural angle

and for holomorphic polar coordinates $\sigma^+_{\hat{x}}$ and $\sigma^-_{\hat{y}}$ associated to a representative $(\hat{x}, \hat{y})$ of $[\hat{x}, \hat{y}]$ there exists $\gamma \in [\gamma]$ such that

$$\tilde{u}^x \circ \sigma^+_{\hat{x}}(s,t) = \big(Ts + c^x, \gamma(kt)\big) + \tilde{r}^x(s,t)$$
$$\tilde{u}^y \circ \sigma^-_{\hat{y}}(s',t') = \big(Ts' + c^y, \gamma(kt')\big) + \tilde{r}^y(s',t')$$

here $\tilde{r}^x, \tilde{r}^y \in H^{3,\delta_0}$

The periodic orbit case

Theorem:

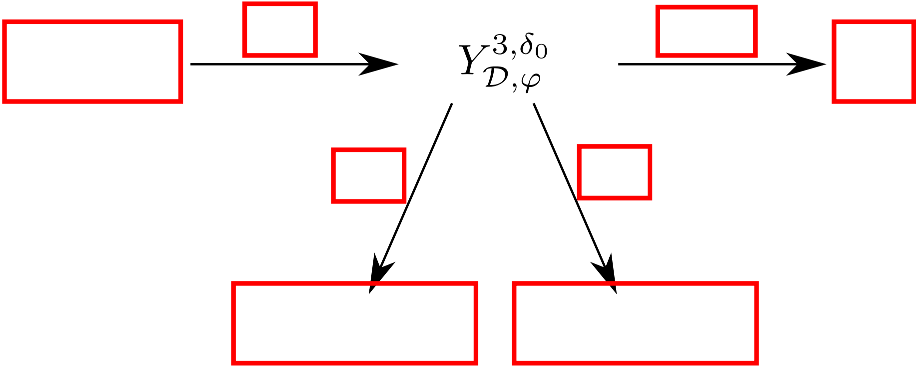
$Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$ is an ssc-Hilbert manifold.

The periodic orbit case

Recall:

$$\begin{array}{ccccc} \mathbb{B}_{\mathcal{D}} \times E^{\delta_0}_{\mathcal{D}} & \xrightarrow{\bigoplus} & X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N) & \xrightarrow{p_{\mathbb{B}_{\mathcal{D}}}} & \mathbb{B}_{\mathcal{D}} \\ & p_x \swarrow & & \searrow p_y & \\ & H^3(A_x, \mathbb{R}^N) & & H^3(A_y, \mathbb{R}^N) & \end{array}$$

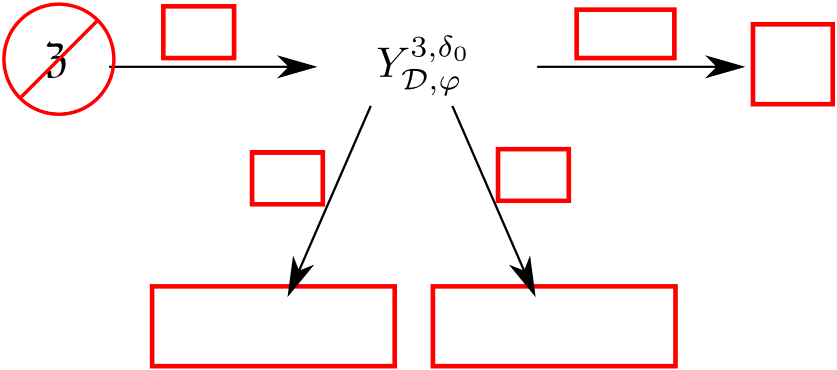
The periodic orbit case



$$Y^{3,\delta_0}_{\mathcal{D},\varphi} = \left(\{0\} \times Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})\right) \bigsqcup$$

$$\left((0,1) \times \bigsqcup_{0<|a|<\frac{1}{4}} H^3(Z_a, \mathbb{R} \times \mathbb{R}^N)\right)$$

The periodic orbit case

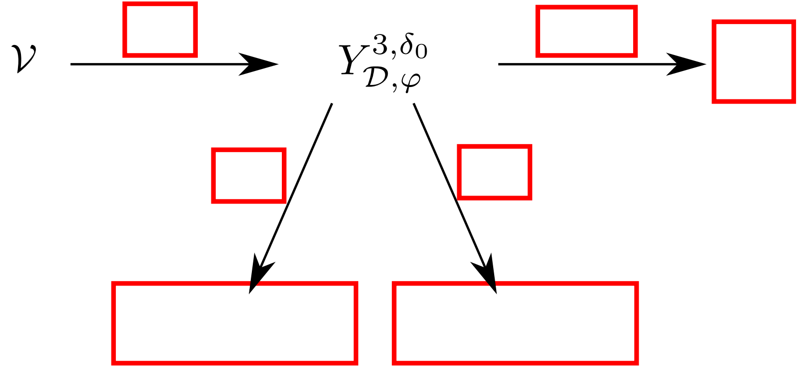


$$\mathfrak{Z} = [0,1) \times Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$$

i.e. elements of the form

$$(r, \tilde{u}) \;\; \text{with } \tilde{u} = (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)$$

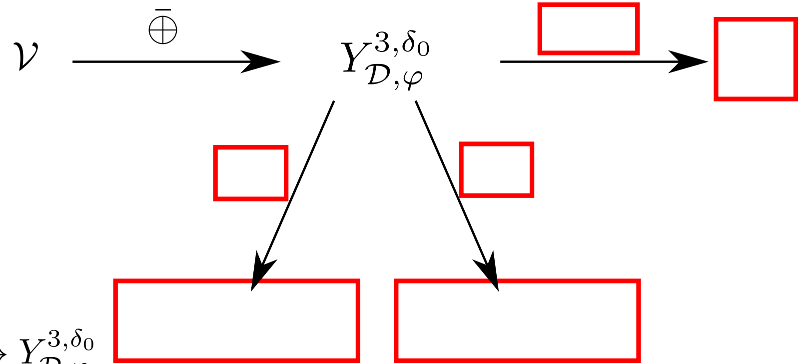
The periodic orbit case



$$\mathcal{V} = \{(r, \tilde{u}) \in \mathfrak{Z} : \text{either } r = 0, \quad \text{or else } \; r > 0 \text{ and } (*) \text{ holds}\}$$

$$(*) \qquad \begin{matrix} \varphi(r) + c^y - c^x > 0 \\ \varphi^{-1}\left(\frac{1}{T} \cdot (\varphi(r) + c^x - c^y)\right) \in (0, \frac{1}{4}) \end{matrix}$$

The periodic orbit case



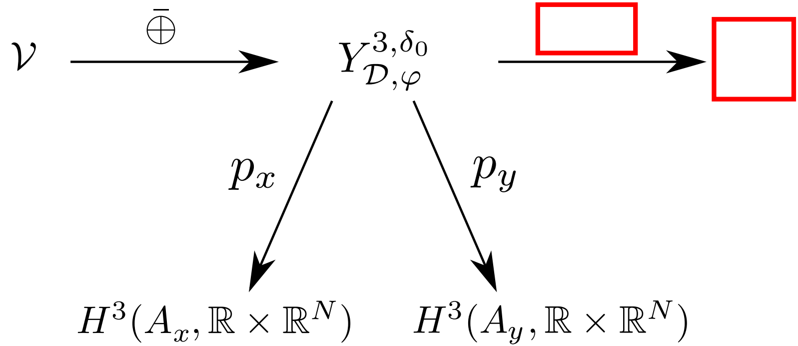
$$\bar{\bigoplus} : \mathcal{V} \to Y^{3,\delta_0}_{\mathcal{D},\varphi}$$

$$\bar{\bigoplus}(0, (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)) = \big(0, (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)\big)$$

$$\bar{\bigoplus}\big(r, (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)\big) = \big(r, \bigoplus_a(\tilde{u}^x, (\varphi(r) * \tilde{u}^y))\big)$$

where $\quad a = |a| \cdot [\hat{x}, \hat{y}] \qquad T \cdot \varphi(|a|) = \varphi(r) + c^y - c^x$

The periodic orbit case



where

$$p_x(r, \tilde{w}) = \tilde{w}\big|_{A_x} \qquad p_y(r, \tilde{w}) = \big((-\varphi(r)) * \tilde{w}\big)\big|_{A_y}$$

The periodic orbit case

$$
\begin{array}{ccccc}
\mathcal{V} & \xrightarrow{\bar{\bigoplus}} & Y^{3,\delta_0}_{\mathcal{D},\varphi} & \xrightarrow{p_{[0,1)}} & [0,1) \\
 & \swarrow^{p_x} & & \searrow^{p_y} & \\
H^3(A_x, \mathbb{R}\times\mathbb{R}^N) & & & & H^3(A_y, \mathbb{R}\times\mathbb{R}^N)
\end{array}
$$

Theorem:

$\left(\bar{\bigoplus}, \{p_x, p_y\}, p_{[0,1)}\right)$ is a ~~subersive imprinting with restrictions.~~ LEGO block.

The periodic orbit case

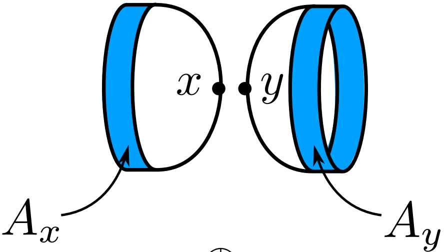
$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\bar{\bigoplus}} & Y^{3,\delta_0}_{\mathcal{D},\varphi} & \xrightarrow{p_{[0,1)}} & [0,1) \\ & \swarrow^{p_x} & & \searrow^{p_y} & \\ H^3(A_x, \mathbb{R}\times\mathbb{R}^N) & & & & H^3(A_y, \mathbb{R}\times\mathbb{R}^N) \end{array}$$

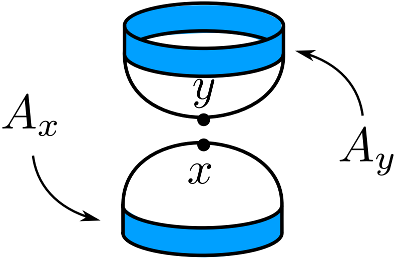
Theorem:

There is a functorial construction which extends to targets $\mathbb{R}\times Q$ from $\mathbb{R}\times\mathbb{R}^N$

Three important cases

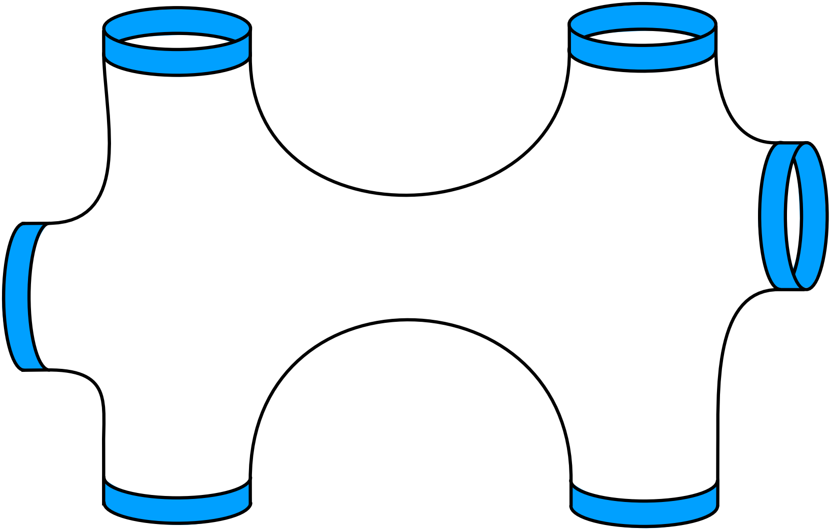
$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\}) \qquad \mathcal{D} = (D_x \sqcup D_y, (x, y))$$





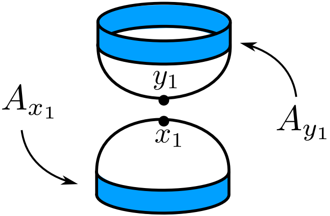
$$\mathbb{B}_{\mathcal{D}} \times E^{\delta_0}_{\mathcal{D}} \xrightarrow{\bigoplus} X^{\delta_0}_{\mathcal{D},\varphi}(\mathbb{R}^N) \qquad \mathcal{V} \xrightarrow{\bar{\bigoplus}} Y^{3,\delta_0}_{\mathcal{D},\varphi}$$

Three important cases

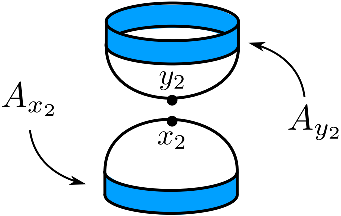


Three important cases

$\mathcal{D}_1 = (D_{x_1} \sqcup D_{y_1}, (x_1, y_1))$



$\mathcal{D}_2 = (D_{x_2} \sqcup D_{y_2}, (x_2, y_2))$

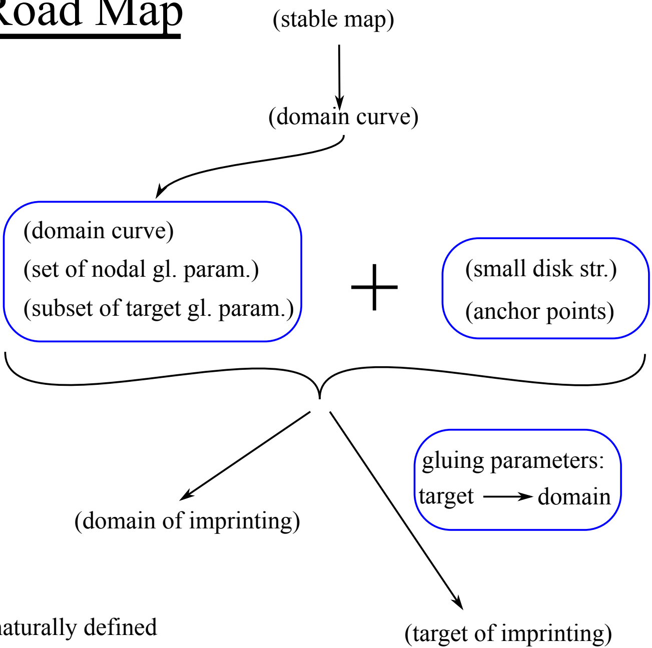


$$\mathcal{V}_1 \xrightarrow{\bar{\oplus}_1} Y^{3,\delta_0}_{\mathcal{D}_1,\varphi} \xrightarrow{p_{[0,1)}} [0,1) \qquad \mathcal{V}_2 \xrightarrow{\bar{\oplus}_2} Y^{3,\delta_0}_{\mathcal{D}_2,\varphi} \xrightarrow{p_{[0,1)}} [0,1)$$

$$\begin{array}{ccccc} \mathcal{V}_1 \times \mathcal{V}_2 & \xrightarrow{\bar{\oplus}_1 \times \bar{\oplus}_2} & Y^{3,\delta_0}_{\mathcal{D}_1,\varphi} \times Y^{3,\delta_0}_{\mathcal{D}_2,\varphi} & \xrightarrow{p_{[0,1)} \times p_{[0,1)}} & [0,1) \times [0,1) \\ & & & & \uparrow \Delta \\ & & & & [0,1) \end{array}$$

Need the pull back of this diagram by Δ

Building -- Road Map



imprinting map naturally defined

Data Preparation:

interface maps

Given:

stable map: $$\alpha = (\alpha_0, \hat{b}_1, \alpha_1, \hat{b}_2, \ldots, \hat{b}_k, \alpha_k)$$

floors

floor: $$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

automorphism group preserving floor structure:

$$G$$

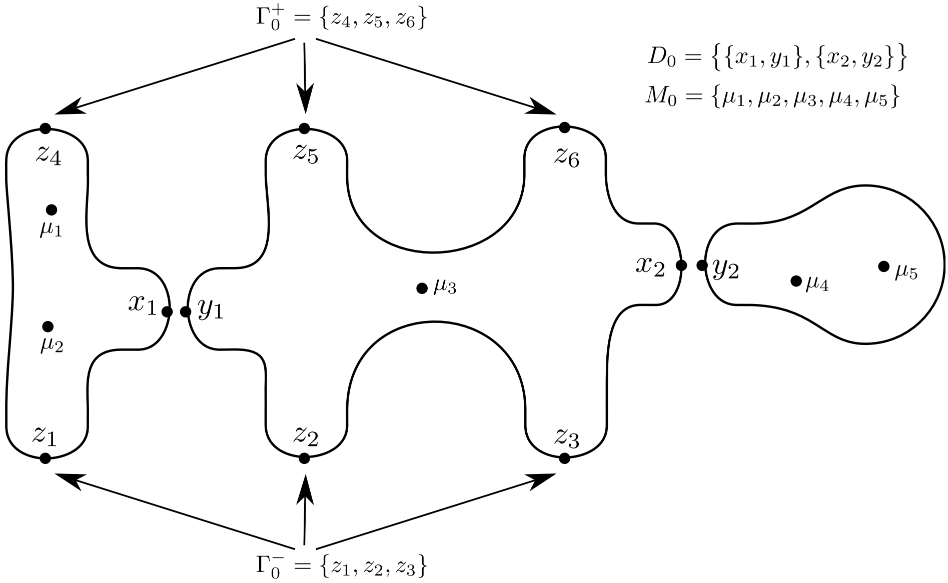
Data Preparation:

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

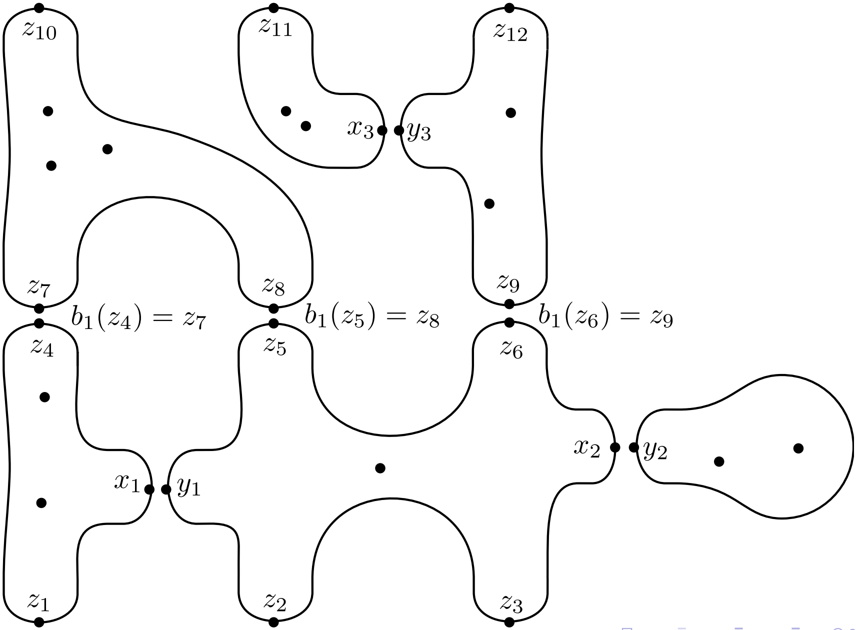
(S_i, j_i)	Riemann surface
M_i	marked points
D_i	nodal pairs
Γ_i^-	negative punctures
Γ_i^+	positive punctures
$[\tilde{u}_i]$	eq. class of maps

$$(a_i, u_i) = \tilde{u}_i \sim c * \tilde{u}_i = (a_i + c, u_i)$$

Data Preparation:



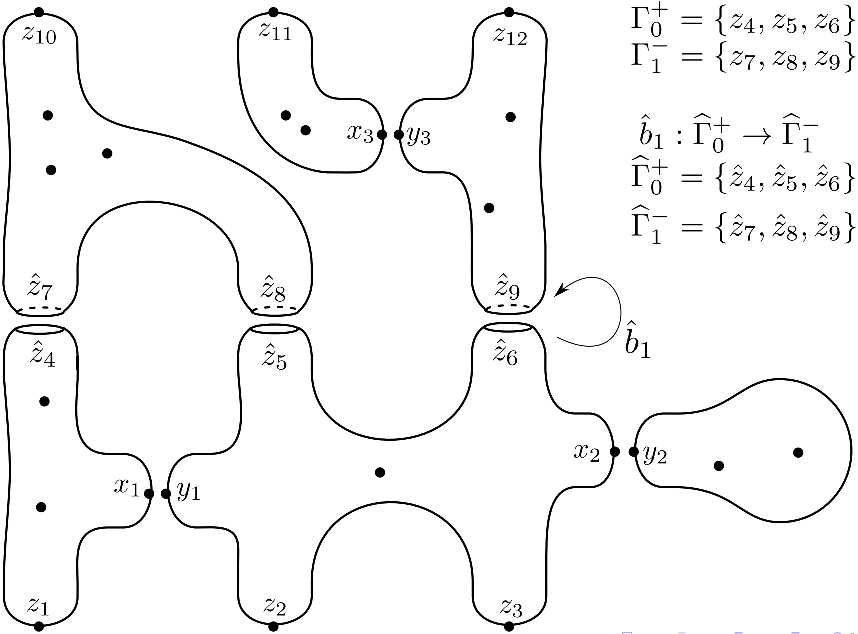
Data Preparation:



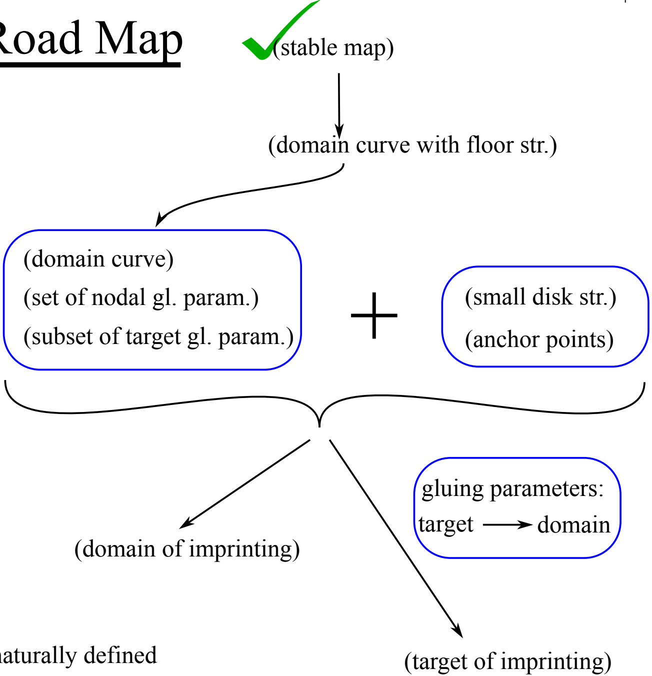
Data Preparation:

$b_1 : \Gamma_0^+ \to \Gamma_1^-$
$\Gamma_0^+ = \{z_4, z_5, z_6\}$
$\Gamma_1^- = \{z_7, z_8, z_9\}$

$\hat{b}_1 : \widehat{\Gamma}_0^+ \to \widehat{\Gamma}_1^-$
$\widehat{\Gamma}_0^+ = \{\hat{z}_4, \hat{z}_5, \hat{z}_6\}$
$\widehat{\Gamma}_1^- = \{\hat{z}_7, \hat{z}_8, \hat{z}_9\}$



Building -- Road Map



imprinting map naturally defined

Data Preparation:

$$\alpha = (\alpha_0, \hat{b}_1, \alpha_1, \hat{b}_2, \dots, \hat{b}_k, \alpha_k)$$

$$\rightsquigarrow$$

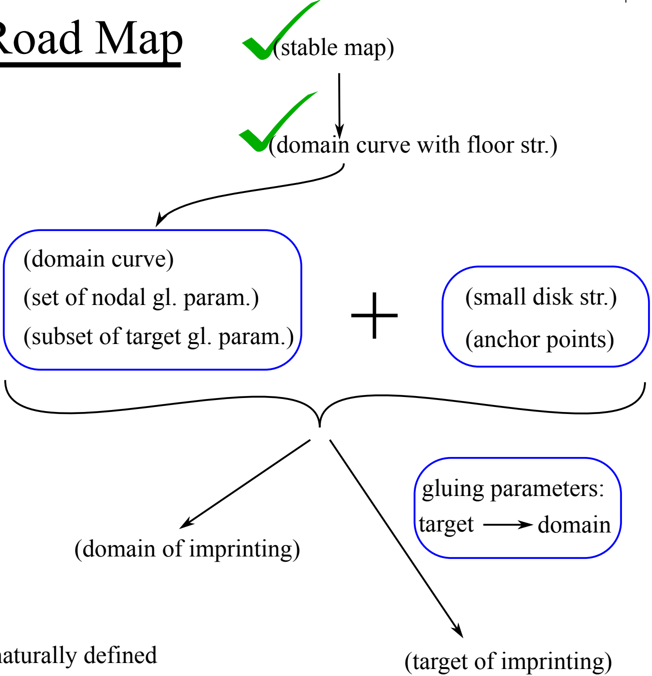
$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \dots, b_k, \sigma_k)$$

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

$$\rightsquigarrow$$

$$\sigma_i = (\Gamma_i^-, S_i, j_i, D_i, \Gamma_i^+)$$

Building -- Road Map



imprinting map naturally defined

Data Preparation:

$$\alpha = (\alpha_0, \hat{b}_1, \alpha_1, \hat{b}_2, \ldots, \hat{b}_k, \alpha_k)$$

$$\downarrow$$

$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \ldots, b_k, \sigma_k)$$

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

$$\downarrow$$

$$\sigma_i = (\Gamma_i^-, S_i, j_i, D_i, \Gamma_i^+)$$

add small disk structures:

$\mathbf{D}_i^+$ about Γ_i^+ $i \in \{0, 1, \ldots, k-1\}$

$\mathbf{D}_i$ about $|D_i|$ $i \in \{0, 1, \ldots, k\}$

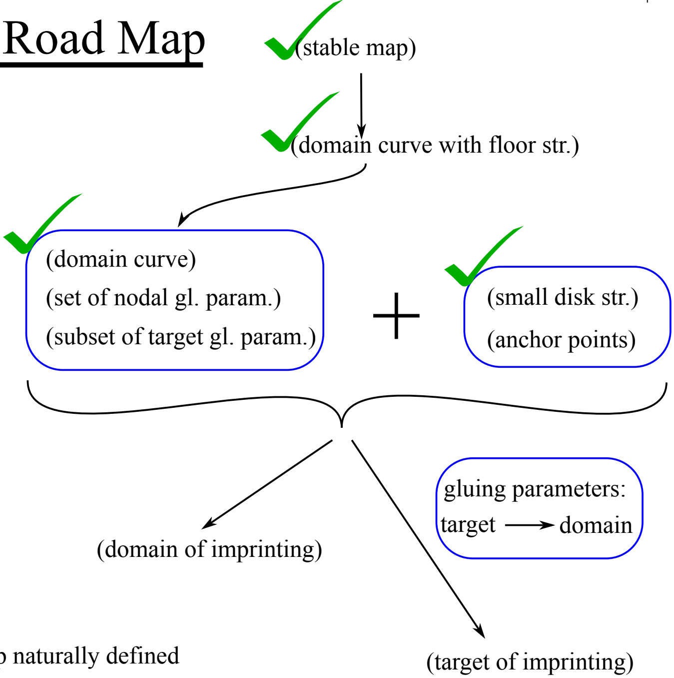
$\mathbf{D}_i^-$ about Γ_i^- $i \in \{1, 2, \ldots, k\}$

add anchor points:

$$\mathcal{J}_i = \mathcal{J} \cap S_i \neq \emptyset$$

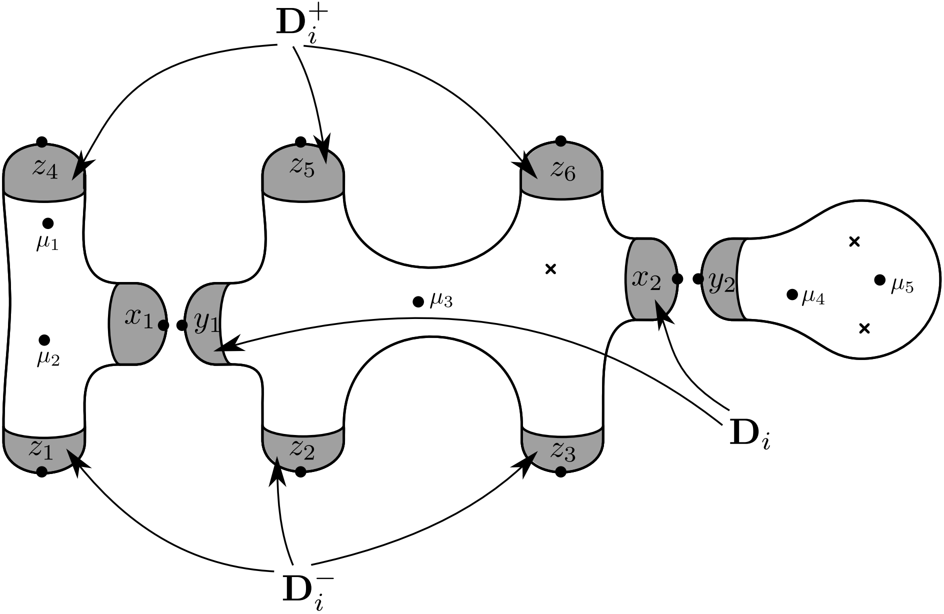
all data G invariant

Building -- Road Map

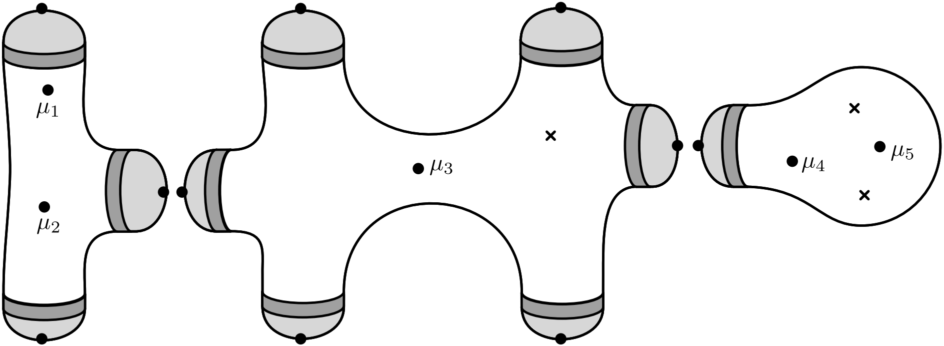


imprinting map naturally defined

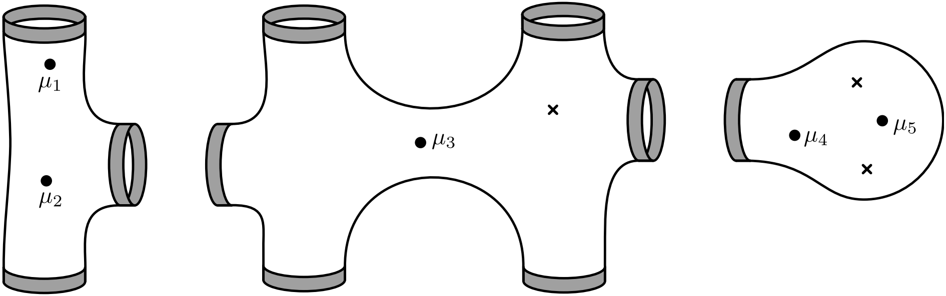
Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor



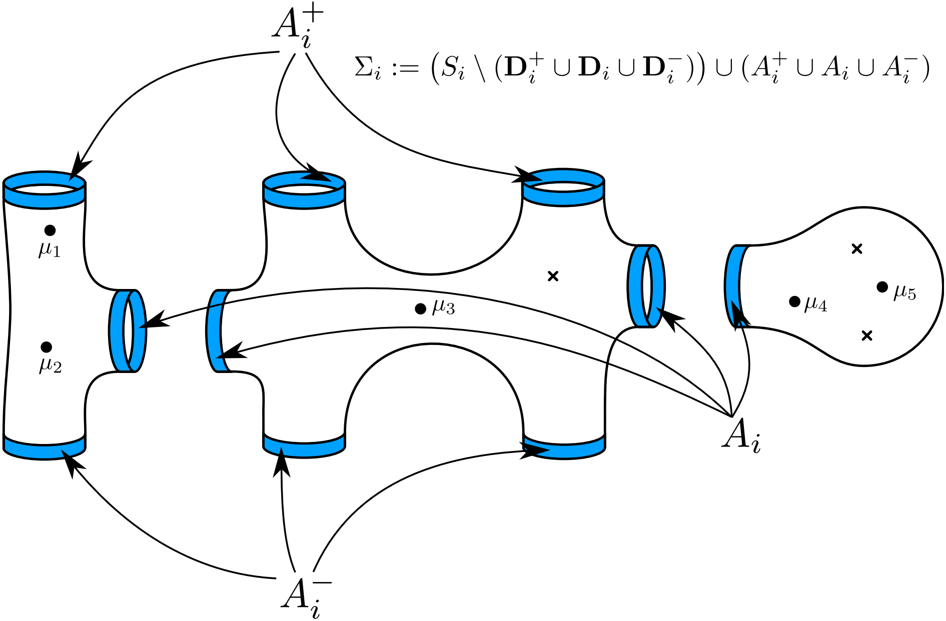
Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor

Fragmentation -- Focus on a floor

$$\Sigma_i := \left(S_i \setminus (\mathbf{D}_i^+ \cup \mathbf{D}_i \cup \mathbf{D}_i^-)\right) \cup (A_i^+ \cup A_i \cup A_i^-)$$



Fragmentation -- Focus on a floor

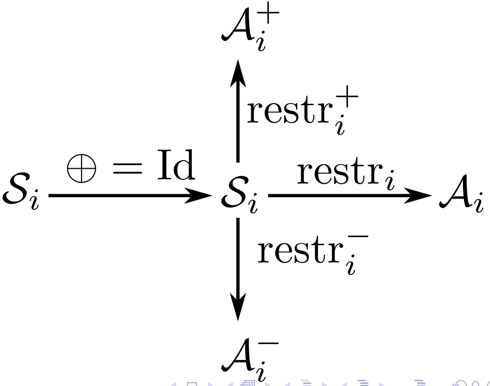
$$\mathcal{S}_i = \left\{ \tilde{u} \in H^3(\Sigma_i) \ : \ \mathrm{av}_{\text{⅄}_i}(\tilde{u}) = 0 \right\}$$

$$\mathcal{A}_i^- = H^3(A_i^-, \mathbb{R} \times \mathbb{R}^N)$$

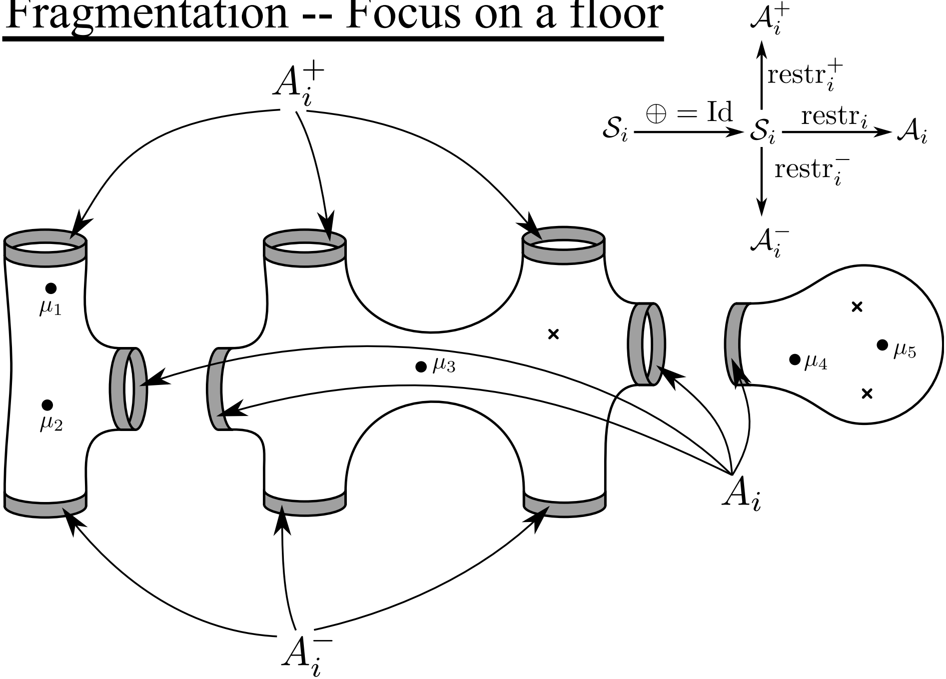
$$\mathcal{A}_i = H^3(A_i, \mathbb{R} \times \mathbb{R}^N)$$

$$\mathcal{A}_i^+ = H^3(A_i^+, \mathbb{R} \times \mathbb{R}^N)$$

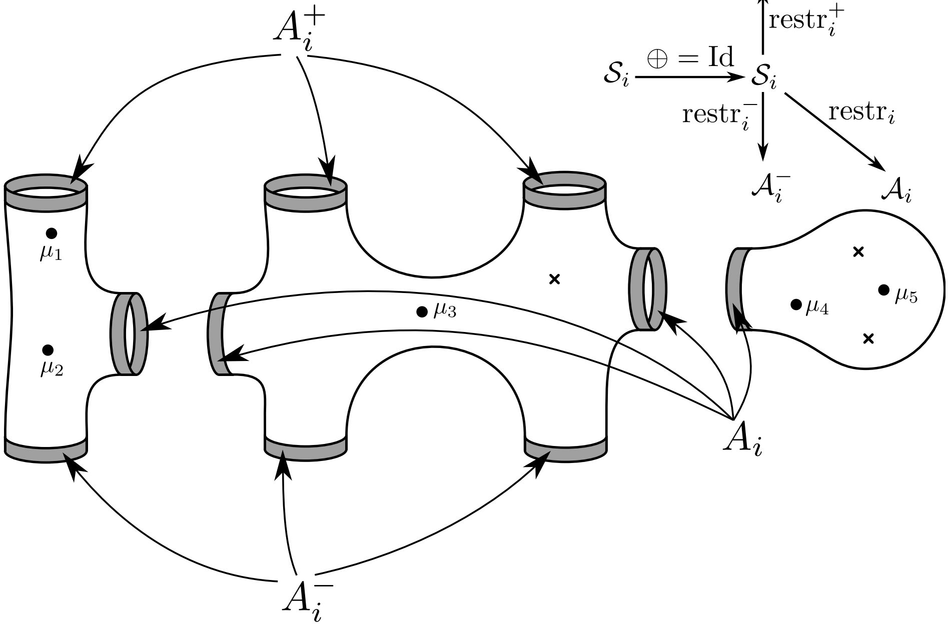
LEGO block



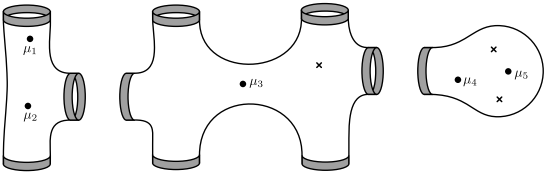
Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor

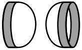


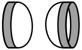
$$\mathcal{S}_i \xrightarrow{\ \oplus\, =\, \mathrm{Id}\ } \mathcal{S}_i$$

$$\mathcal{S}_i \xrightarrow{\mathrm{restr}_i^+} \mathcal{A}_i^+ \qquad \mathcal{S}_i \xrightarrow{\mathrm{restr}_i^-} \mathcal{A}_i^- \qquad \mathcal{S}_i \xrightarrow{\mathrm{restr}_i} \mathcal{A}_i$$

$$(\mathbb{B}_{\mathcal{D}_1}\times E^{\delta_0}_{\mathcal{D}_1})\times(\mathbb{B}_{\mathcal{D}_2}\times E^{\delta_0}_{\mathcal{D}_2}) \xrightarrow{\oplus_1\times\oplus_2} X^{\delta_0}_{\mathcal{D}_1,\varphi}\times X^{\delta_0}_{\mathcal{D}_2,\varphi} \xrightarrow{p_{\mathbb{B}_{\mathcal{D}_1}}\times p_{\mathbb{B}_{\mathcal{D}_2}}} \mathbb{B}_{\mathcal{D}_1}\times\mathbb{B}_{\mathcal{D}_2}$$

$$X^{\delta_0}_{\mathcal{D}_1,\varphi}\times X^{\delta_0}_{\mathcal{D}_2,\varphi} \xrightarrow{\mathrm{restr}_i} \mathcal{A}_i$$



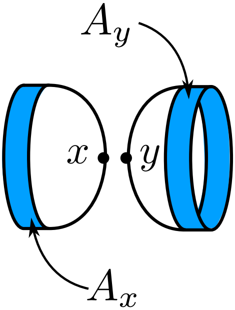


Fragmentation -- Construct a building

Recall: $\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$

$$\mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \xrightarrow{\bigoplus} X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \xrightarrow{p_{\mathbb{B}_{\mathcal{D}}}} \mathbb{B}_{\mathcal{D}}$$

$$p_x : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_x, \mathbb{R}^N), \qquad p_y : X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \to H^3(A_y, \mathbb{R}^N)$$



$$\underline{\underline{E_{\mathcal{D}}^{\delta_0}}} = \mathbb{R}^N \oplus H^{3,\delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

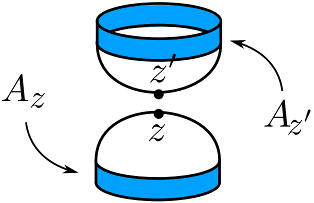
Fragmentation -- Construct a building

Recall:

$$\mathcal{V} \xrightarrow{\bar{\bigoplus}} Y^{3,\delta_0}_{\mathcal{D},\varphi} \xrightarrow{p_{[0,1)}} [0,1)$$

$$Y^{3,\delta_0}_{\mathcal{D},\varphi} \xrightarrow{p_z} H^3(A_z, \mathbb{R} \times \mathbb{R}^N) \qquad Y^{3,\delta_0}_{\mathcal{D},\varphi} \xrightarrow{p_{z'}} H^3(A_{z'}, \mathbb{R} \times \mathbb{R}^N)$$

$\mathcal{D} = (D_z \sqcup D_{z'}, (z, z'))$



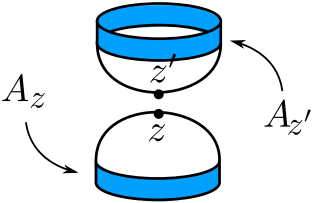
Fragmentation -- Construct a building

Recall:

$$\mathcal{V} \xrightarrow{\bar{\bigoplus}} Y^{3,\delta_0}_{\mathcal{D},\varphi} \xrightarrow{p_{[0,1)}} [0,1)$$

$$p_z : Y^{3,\delta_0}_{\mathcal{D},\varphi} \to H^3(A_z, \mathbb{R} \times \mathbb{R}^N) \qquad p_{z'} : Y^{3,\delta_0}_{\mathcal{D},\varphi} \to H^3(A_{z'}, \mathbb{R} \times \mathbb{R}^N)$$

$\mathcal{D} = (D_z \sqcup D_{z'}, (z, z'))$



$\mathcal{V} \subset [0,1) \times Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$

elements of the form $\tilde{u} = (\tilde{u}^z, [\hat{x}, \hat{y}], \tilde{u}^{z'})$

$$\tilde{u}^z \circ \sigma^+_{\hat{x}}(s,t) = \big(Ts + c^z, \gamma(kt)\big) + \tilde{r}^z(s,t)$$

$$\tilde{u}^{z'} \circ \sigma^-_{\hat{y}}(s',t') = \big(Ts' + c^{z'}, \gamma(kt')\big) + \tilde{r}^{z'}(s',t')$$

here $\tilde{r}^z, \tilde{r}^{z'} \in H^{3,\delta_0}$

Recall:

$$\Sigma_i := \big(S_i \setminus (\mathbf{D}_i^+ \cup \mathbf{D}_i \cup \mathbf{D}_i^-)\big) \cup (A_i^+ \cup A_i \cup A_i^-)$$

$$\mathcal{S}_i := \big\{\tilde{u} \in H^3(\Sigma_i) \ : \ \mathrm{av}_{\daleth_i}(\tilde{u}) = 0\big\}$$

$$\mathrm{av}_{\daleth_i}(\tilde{u}) := \frac{1}{\#\daleth_i} \cdot \sum_{z \in \daleth_i} a_i(z)$$

$$\Gamma := \bigcup_{i=0}^{k} (\Gamma_i^+ \cup \Gamma_i^-)$$

$$\overline{\mathbf{F}} : \Gamma \to \big\{\bar{\boldsymbol{\gamma}} \ : \ \text{weighted periodic orbit in } \mathbb{R}^N$$

which satisfies $\overline{\mathbf{F}}(z) = \overline{\mathbf{F}}(b_i(z))$ for each $z \in \Gamma_i^+$ and $i \in \{0, \ldots, k-1\}\big\}$

Define ssc-Hilbert manifold $Z^3_{\sigma,\daleth}(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}})$

$Z^3_{\sigma,\lambda}(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}}) =$

	$Z^-_{\mathcal{D}_0} \times_{\mathcal{A}^-_0}$	Negative ends of bottom level
(truncated) floor 0	$\mathcal{S}_0 \times_{\mathcal{A}_0} E_{\mathcal{D}_0}$	
	$\times_{\mathcal{A}^+_0} Z_{\mathcal{D}_1} \times_{\mathcal{A}^-_1}$	Interface level 1
(truncated) floor 1	$\mathcal{S}_1 \times_{\mathcal{A}_1} E_{\mathcal{D}_1}$	
	$\times_{\mathcal{A}^+_1} Z_{\mathcal{D}_2} \times_{\mathcal{A}^-_2}$	Interface level 2
(truncated) floor 2	$\mathcal{S}_2 \times_{\mathcal{A}_2} E_{\mathcal{D}_2}$	
	$\times_{\mathcal{A}^+_2} Z_{\mathcal{D}_3} \times_{\mathcal{A}^-_3}$	Interface level 3
	$\vdots$	
	$\times_{\mathcal{A}^+_{k-1}} Z_{\mathcal{D}_k} \times_{\mathcal{A}^-_k}$	Interface level k
(trucated) floor k	$\mathcal{S}_k \times_{\mathcal{A}_k} E_{\mathcal{D}_k}$	
	$\times_{\mathcal{A}^+_k} Z^+_{\mathcal{D}_k}$	Positive ends of top level

The takeaway:

$Z^3_{\sigma,\lambda}(\mathbb{R}\times\mathbb{R}^N,\overline{\mathbf{F}})$ is an ssc-manifold consisting of tuples of the form

$$\tilde{u} := (\tilde{u}_0, \hat{b}_1, \ldots, \hat{b}_k, \tilde{u}_k)$$

where each $\tilde{u}_i$ is of class $(3, \delta_0)$ and asymptotic to the weighted periodic orbits prescribed by $\overline{\mathbf{F}}$ so that the data across interfaces is $\hat{b}_i$ matching, and the anchor averages vanish.

Domain of Imprinting (almost):

ssc-manifold just constructed

$$\mathbb{B}_{\mathcal{D}} \times [0,1)^k \times Z^3_{\sigma,\lambda}(\mathbb{R}\times\mathbb{R}^N, \overline{\mathbf{F}})$$

tuple of target gluing parameters one for each interface level

product of domain gluing parameters $\mathbb{B}_{\mathcal{D}_{\{x,y\}}}$ for each $\{x,y\} \in D$

Domain of Imprinting (actual):

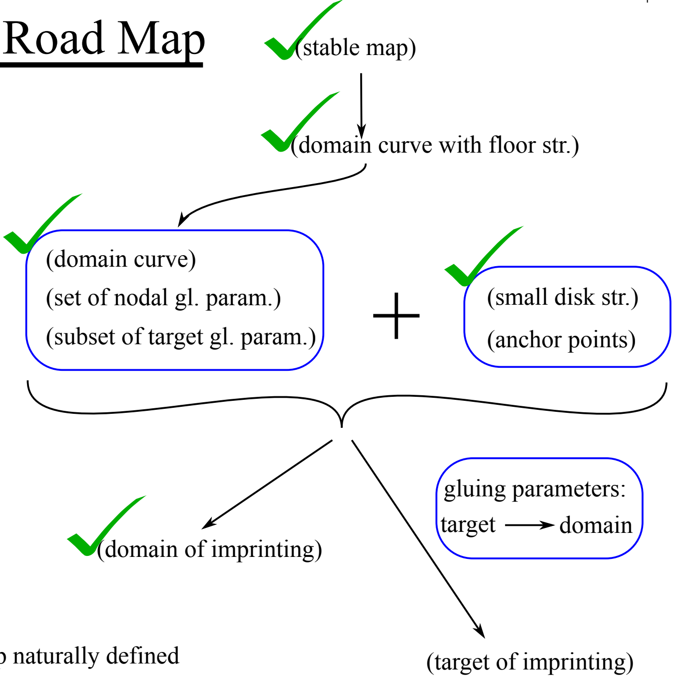
$$\mathbb{B}_{\mathcal{D}} \times \mathcal{O}$$

where $\mathcal{O} \subset [0,1)^k \times Z^3_{\sigma,\rotatebox{180}{λ}}(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}})$ consists of tuples $(r_1, \ldots, r_k, \tilde{u})$ with $r_i \in (0,1)$ and $\tilde{u} := (\tilde{u}_0, \hat{b}_1, \ldots, \hat{b}_k, \tilde{u}_k)$ such that either

1. $r_i = 0$

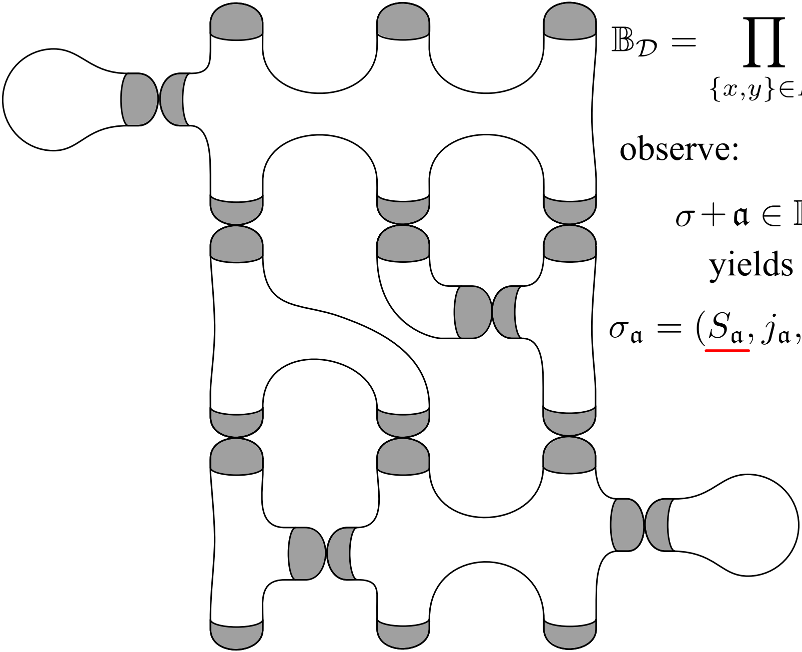
2. $$\begin{cases} \varphi(r_i) - c^z(\tilde{u}) + c^{b_i(z)}(\tilde{u}) > 0 \\ \varphi^{-1}\left(\frac{1}{T_z} \cdot (\varphi(r_i) - c^z(\tilde{u}) + c^{b_i(z)}(\tilde{u})\right) \in (0, \frac{1}{4}) \end{cases}$$

Building -- Road Map



imprinting map naturally defined

$\sigma = (S, j, M, D)$



Let

$$\mathbb{B}_{\mathcal{D}} = \prod_{\{x,y\} \in D} \mathbb{B}_{\mathcal{D}_{\{x,y\}}}$$

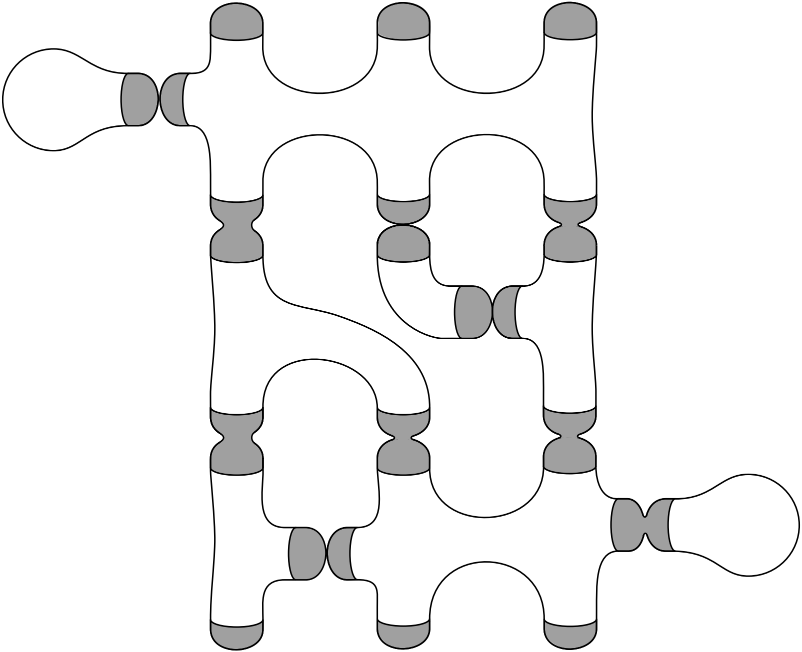
observe:

$$\sigma + \mathfrak{a} \in \mathbb{B}_{\mathcal{D}}$$

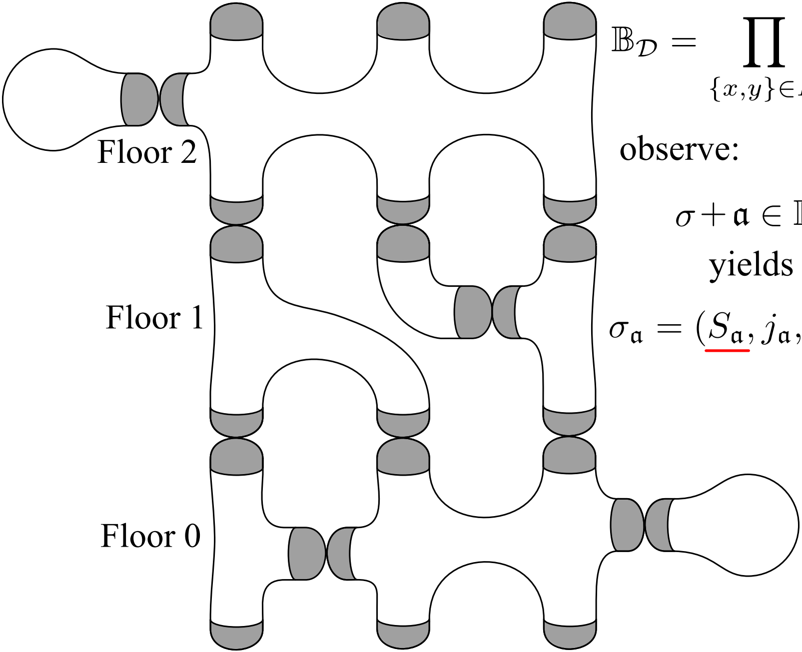
yields

$$\sigma_{\mathfrak{a}} = (\underline{S_{\mathfrak{a}}}, j_{\mathfrak{a}}, M_{\mathfrak{a}}, \underline{D_{\mathfrak{a}}})$$

$\sigma_{\mathfrak{a}} = (S_{\mathfrak{a}}, j_{\mathfrak{a}}, M_{\mathfrak{a}}, D_{\mathfrak{a}})$ $\mathfrak{a} \in \mathbb{B}_{\mathcal{D}}$



$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \sigma_2)$$



Let

$$\mathbb{B}_{\mathcal{D}} = \prod_{\{x,y\}\in D} \mathbb{B}_{\mathcal{D}_{\{x,y\}}}$$

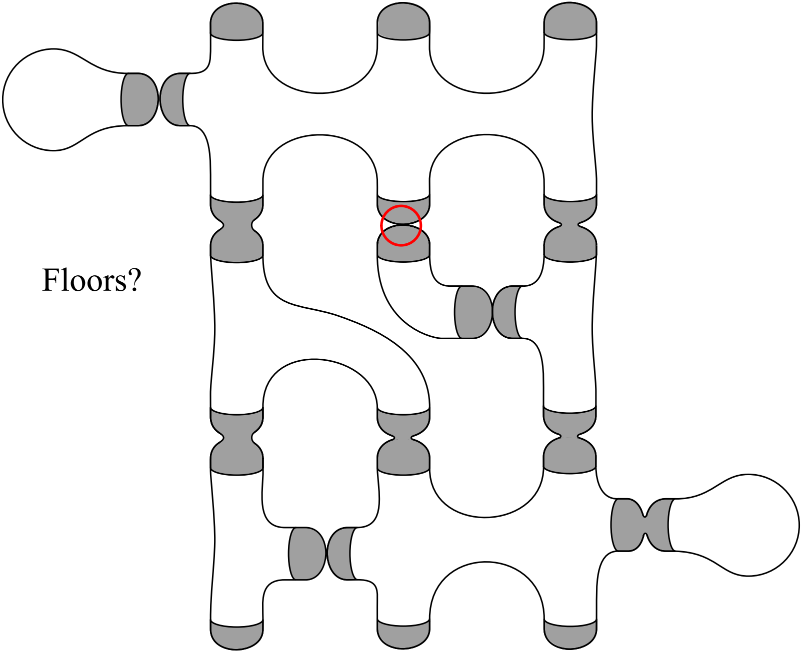
observe:

$$\sigma + \mathfrak{a} \in \mathbb{B}_{\mathcal{D}}$$

yields

$$\sigma_{\mathfrak{a}} = (\underline{S_{\mathfrak{a}}}, j_{\mathfrak{a}}, M_{\mathfrak{a}}, \underline{D_{\mathfrak{a}}})$$

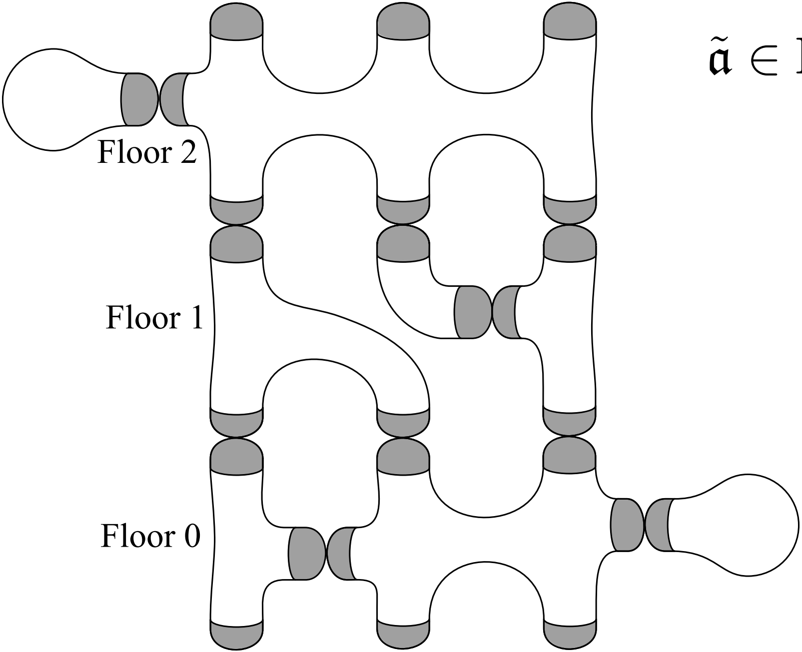
$\sigma_{\mathfrak{a}} = \text{nonsense}$ $\mathfrak{a} \in \mathbb{B}_{\mathcal{D}}$



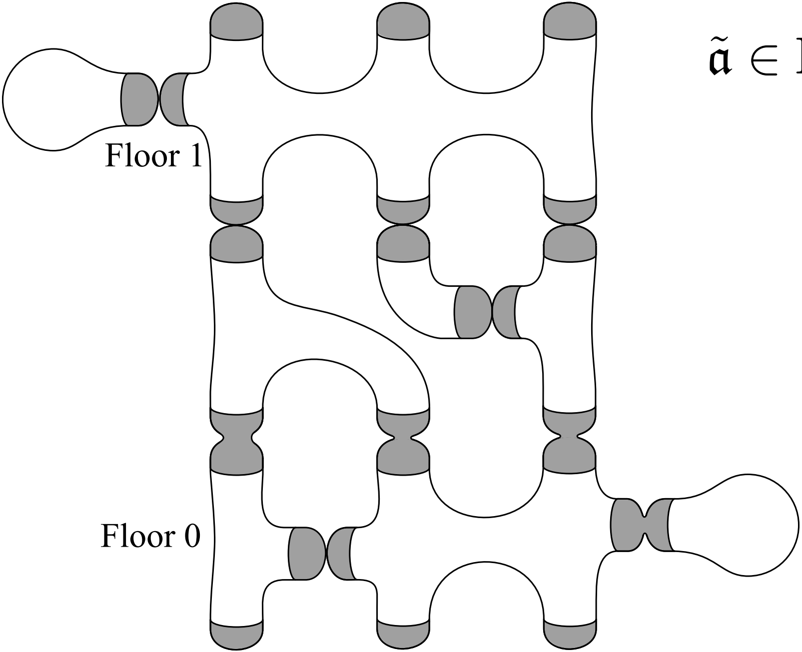
Floors?

$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \sigma_2)$

$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$

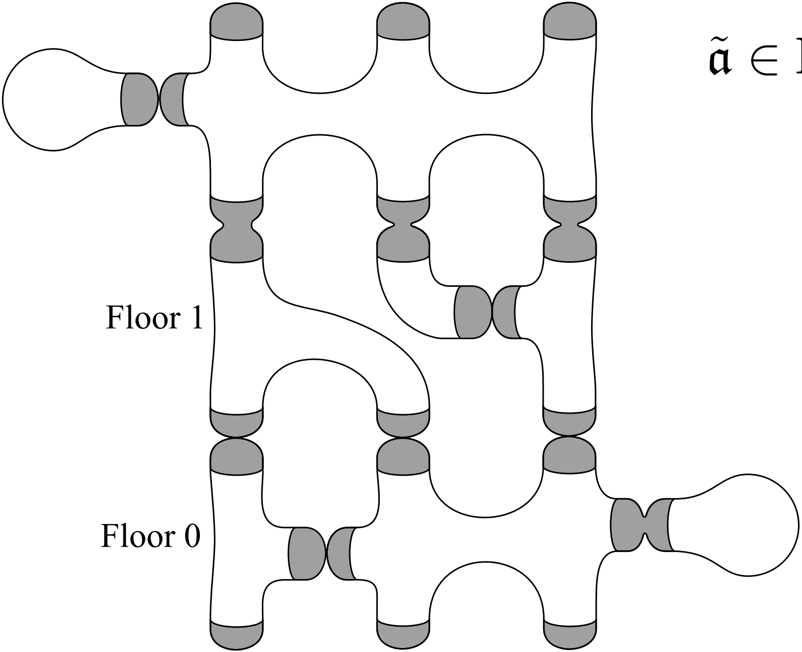


$$\sigma_{\tilde{\mathfrak{a}}} = (\sigma_0^{\tilde{\mathfrak{a}}}, b_1^{\tilde{\mathfrak{a}}}, \sigma_1^{\tilde{\mathfrak{a}}})$$



$$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$$

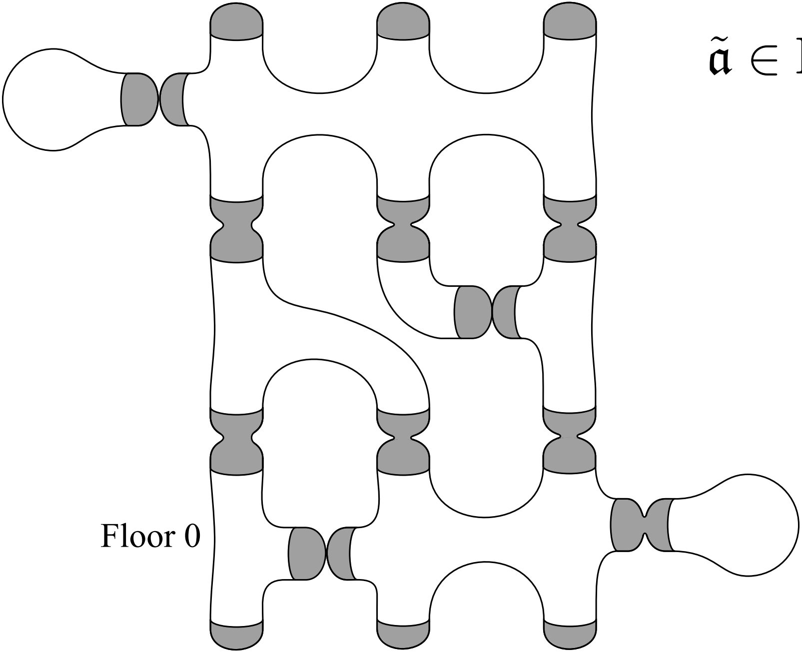
$$\sigma_{\tilde{\mathfrak{a}}} = (\sigma_0^{\tilde{\mathfrak{a}}}, b_1^{\tilde{\mathfrak{a}}}, \sigma_1^{\tilde{\mathfrak{a}}})$$



$$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$$

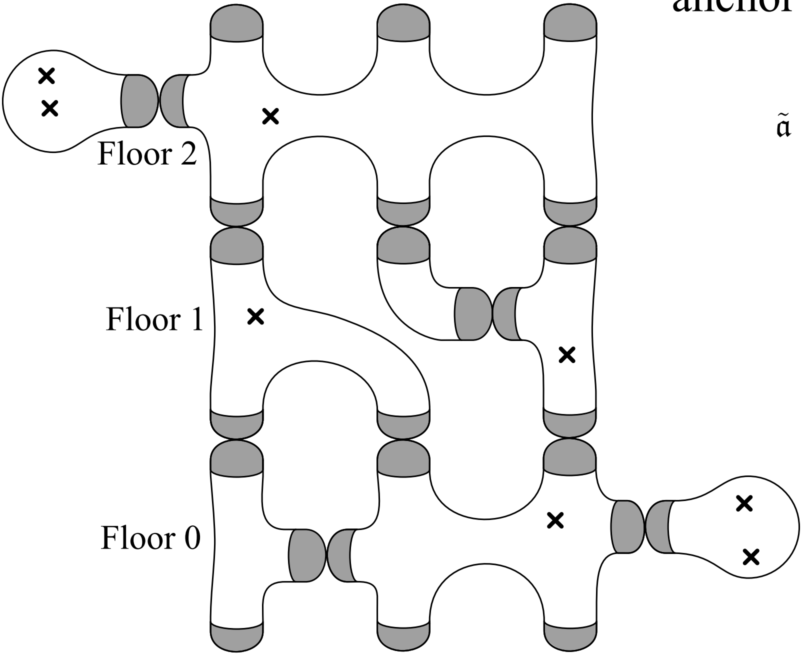
$\sigma_{\tilde{\mathfrak{a}}} = (\sigma_0^{\tilde{\mathfrak{a}}})$

$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$



$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \sigma_2)$

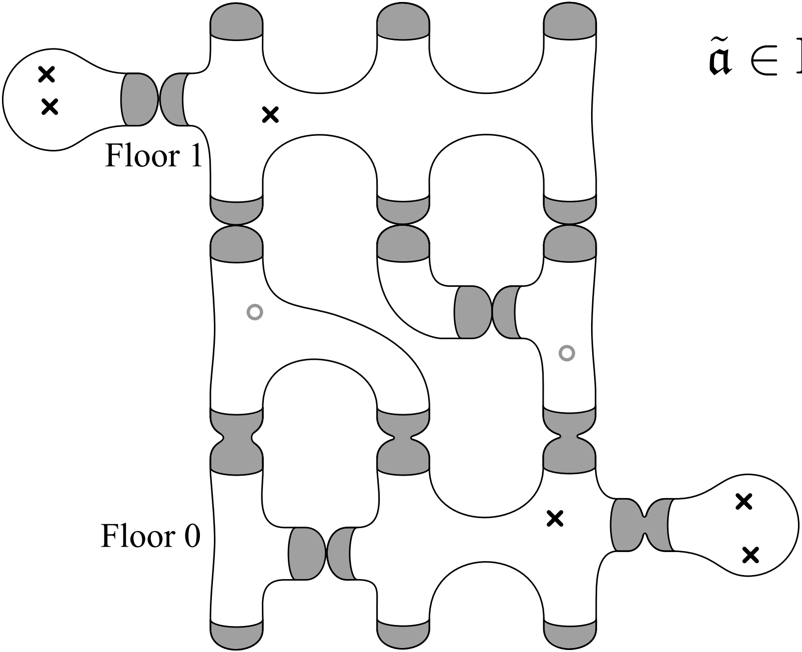
anchor points



$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$

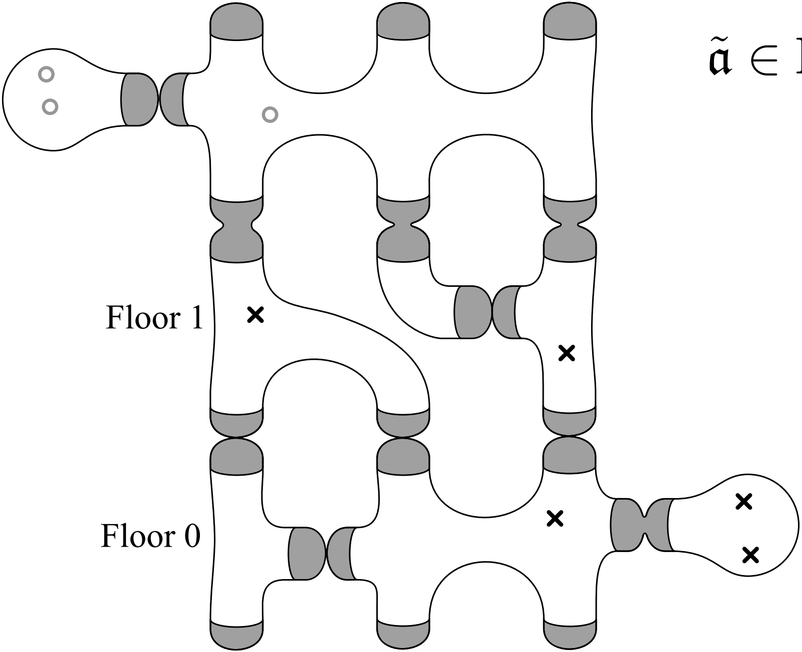
$$\sigma_{\tilde{\mathfrak{a}}} = (\sigma_0^{\tilde{\mathfrak{a}}}, b_1^{\tilde{\mathfrak{a}}}, \sigma_1^{\tilde{\mathfrak{a}}})$$

$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$



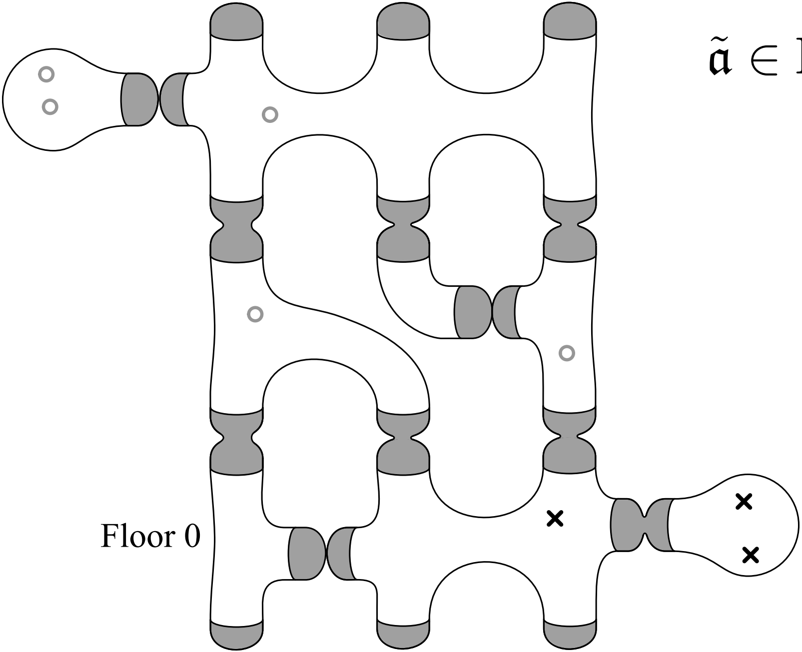
$$\sigma_{\tilde{\mathfrak{a}}} = (\sigma_0^{\tilde{\mathfrak{a}}}, b_1^{\tilde{\mathfrak{a}}}, \sigma_1^{\tilde{\mathfrak{a}}})$$

$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$



$\sigma_{\tilde{\mathfrak{a}}} = (\sigma_0^{\tilde{\mathfrak{a}}})$

$\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}$



Recall: $\sigma_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, \Gamma_i^+)$

up to rearrangement: $\alpha = \big((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, ([\tilde{u}_i])_{i=0}^k\big)$

$$\big((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, ([\tilde{u}_i])_{i=0}^k\big)$$

$$\Big\downarrow \; +(\gimel_i)_{i=0}^k$$

$$\big((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, \underline{(\tilde{u}_i)_{i=0}^k}, (\gimel_i)_{i=0}^k\big)$$

$+$ $(r_i)_{i=1}^k \in [0,1)^k$ and $\mathbf{a} \in \mathbb{B}_{\mathcal{D}}$

conditioned on being in $\mathbb{B}_{\mathcal{D}} \times \mathcal{O}$

$$\hat{b}_i\big|_z \rightsquigarrow [\hat{x}, \hat{y}]_{(z, b_i(z))}$$

$$(z, \tilde{u}_i, r_i) \rightsquigarrow |a_{(z, b_i(z))}| \quad \text{via}$$

$$T_{\mathbf{F}(z,b_i(z))} \cdot \varphi(|a_{(z,b_i(z))}|) = \varphi(r_i) - c^z(\tilde{u}_i) + c^{b_i(z)}(\tilde{u}_i)$$

$$\left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, ([\tilde{u}_i])_{i=0}^k\right)$$

$$\Big\downarrow \; + (\text{J}_i)_{i=0}^k$$

$$\left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, \underline{(\tilde{u}_i)_{i=0}^k}, (\text{J}_i)_{i=0}^k\right)$$

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$$T_{\mathbf{F}(z,b_i(z))} \cdot \varphi(|a_{(z,b_i(z))}|) = \varphi(r_i) - c^z(\tilde{u}_i) + c^{b_i(z)}(\tilde{u}_i)$$

$$\rightsquigarrow |a| \cdot [\hat{x}, \hat{y}] = a$$

$$\Big\downarrow$$

$$\left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, (\tilde{u}_i)_{i=0}^k, (\text{J}_i)_{i=0}^k, \underline{\tilde{\mathfrak{a}} \in \mathbb{B}_{\text{ad}}}\right)$$

$$\Big|$$

$$\left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, \underline{(\tilde{u}_i)_{i=0}^k}, (\daleth_i)_{i=0}^k\right)$$

$\downarrow$ $+$ $(r_i)_{i=1}^k \in [0,1)^k$ and $\mathbf{a} \in \mathbb{B}_{\mathcal{D}}$

conditioned on being in $\mathbb{B}_{\mathcal{D}} \times \mathcal{O}$

$$\hat{b}_i\big|_z \rightsquigarrow [\hat{x}, \hat{y}]_{(z, b_i(z))}$$

$$(z, \tilde{u}_i, r_i) \rightsquigarrow |a_{(z,b_i(z))}| \quad \text{via}$$

$$T_{\mathbf{F}(z,b_i(z))} \cdot \varphi(|a_{(z,b_i(z))}|) = \varphi(r_i) - c^z(\tilde{u}_i) + c^{b_i(z)}(\tilde{u}_i)$$

$$\rightsquigarrow |a| \cdot [\hat{x}, \hat{y}] = a$$

$$\left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, (\tilde{u}_i)_{i=0}^k, (\daleth_i)_{i=0}^k, \underline{\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}}\right)$$

$\downarrow$

$$\left(\underline{(\sigma_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, (\daleth_{\tilde{\mathfrak{a}},e})_{e=0}^\ell}, (\tilde{u}_i)_{i=0}^k, (\daleth^{\mathrm{vir}}_{\tilde{\mathfrak{a}},i})_{i=0}^k, \tilde{\mathfrak{a}}\right)$$

for $i \leq i < i_{+1}$

$$\Big((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, (\tilde{u}_i)_{i=0}^k, (\text{Ʝ}_i)_{i=0}^k, \underline{\tilde{\mathfrak{a}} \in \mathbb{B}_{\mathrm{ad}}}\Big)$$

$$\downarrow$$

$$\Big(\underline{(\sigma_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, (\text{Ʝ}_{\tilde{\mathfrak{a}},e})_{e=0}^\ell}, (\tilde{u}_i)_{i=0}^k, (\text{Ʝ}^{\mathrm{vir}}_{\tilde{\mathfrak{a}},i})_{i=0}^k, \tilde{\mathfrak{a}}\Big)$$

$$\downarrow$$

for $i_e \le i < i_{e+1}$

$$\tilde{u}_i^* = \begin{cases} \tilde{u}_i & \text{if } i = i_e \\ \big(\varphi(r_{i_e+1}) + \cdots + \varphi(r_i)\big) * \tilde{u}_i & \text{otherwise} \end{cases}$$

$$\tilde{w}_e = \oplus_{\tilde{\mathfrak{a}}_e}(\tilde{u}_{i_e}^*, \ldots, \tilde{u}_{i_{e+1}-1}^*)$$

$$\Big(\underbrace{(\sigma_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, \underline{(\tilde{w}_e)_{e=0}^\ell}}_{\in\ Z^{3,\delta_0}_{\boldsymbol{\sigma},\text{Ʝ},\varphi}(\mathbb{R}\times\mathbb{R}^N, \mathbf{F})}, (\text{Ʝ}_{\tilde{\mathfrak{a}},e})_{e=0}^\ell, (\text{Ʝ}^{\mathrm{vir}}_{\tilde{\mathfrak{a}},i})_{i=0}^k, \tilde{\mathfrak{a}}\Big)$$

$$\Big((\sigma_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell}, (\hat{b}_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell}, (\daleth_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell}, (\tilde{u}_i)_{i=0}^{k}, (\daleth^{\mathrm{vir}}_{\tilde{\mathfrak{a}},i})_{i=0}^{k}, \tilde{\mathfrak{a}}\Big)$$

$$\Big\downarrow$$

for $i_e \le i < i_{e+1}$

$$\tilde{u}_i^* = \begin{cases} \tilde{u}_i & \text{if } i = i_e \\ \big(\varphi(r_{i_e+1}) + \cdots + \varphi(r_i)\big) * \tilde{u}_i & \text{otherwise} \end{cases}$$

$$\tilde{w}_e = \oplus_{\tilde{\mathfrak{a}}_e}(\tilde{u}_{i_e}^*, \dots, \tilde{u}_{i_{e+1}-1}^*)$$

$$\underbrace{\Big((\sigma_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell}, (\hat{b}_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell}, (\tilde{w}_e)_{e=0}^{\ell}}, (\daleth_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell}, (\daleth^{\mathrm{vir}}_{\tilde{\mathfrak{a}},i})_{i=0}^{k}, \tilde{\mathfrak{a}}\Big)$$

$$\in \; Z^{3,\delta_0}_{\boldsymbol{\sigma},\daleth,\varphi}(\mathbb{R} \times \mathbb{R}^N, \mathbf{F})$$

$$\underbrace{\Big((\sigma_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell},(\hat{b}_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell},\underline{(\tilde{w}_e)_{e=0}^{\ell}}}_{\in\; Z^{3,\delta_0}_{\boldsymbol{\sigma},\text{\gimel},\varphi}(\mathbb{R}\times\mathbb{R}^N,\mathbf{F})},(\text{\gimel}_{\tilde{\mathfrak{a}},e})_{e=0}^{\ell},(\text{\gimel}^{\text{vir}}_{\tilde{\mathfrak{a}},i})_{i=0}^{k},\tilde{\mathfrak{a}}\Big)$$

Workhorse Imprinting Theorem:

This defines an imprinting

$$\overline{\bigoplus} : \mathbb{B}_D \times \mathcal{O} \to Z^3_{\boldsymbol{\sigma},\daleth,\varphi}(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}})$$

Moreover this functorially extends to an imprinting for

$$Z^3_{\boldsymbol{\sigma},\daleth,\varphi}(\mathbb{R} \times Q, \overline{\mathbf{F}})$$

Building -- Road Map

(stable map)

(domain curve with floor str.)

(domain curve)
(set of nodal gl. param.)
(subset of target gl. param.)

+

(small disk str.)
(anchor points)

gluing parameters:
target ⟶ domain

(domain of imprinting)

(target of imprinting)

imprinting map naturally defined

Transversal Constraints:

Consider a map in $Z^3_{\boldsymbol{\sigma},\daleth,\varphi}(\mathbb{R}\times Q,\overline{\mathbf{F}})$ fix a G invariant finite set $\Xi=\Xi_0\cup\ldots\cup\Xi_k$ disjoint from usual interesting sets. For $z\in\Xi_i$ let $[z]$ denote its G orbit. There are two types of constraints:

- $\mathbb{R}$ invariant:

  Fix co-dimension 2 submanifold $H_{[z]}\subset Q$

$$\widetilde{H}_{[z]}:=\mathbb{R}\times H_{[z]}$$

- non $\mathbb{R}$ invariant:

  Fix co-dimension 1 submanifold $H_{[z]}\subset Q$

$$\widetilde{H}_{[z]}:=\{\bar{a}_{[z]}\}\times H_{[z]}$$

Transversal Constraints:

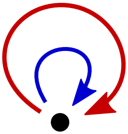
This yields an assignment: $\mathcal{H} : z \mapsto \widetilde{H}_{[z]}$

Then define the subset $Z^3_{\boldsymbol{\sigma},\gimel,\mathcal{H},\varphi}(\mathbb{R} \times Q, \overline{\mathbf{F}})$ of

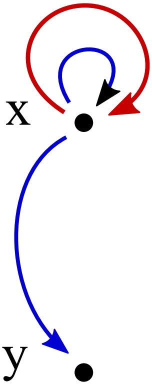
$Z^3_{\boldsymbol{\sigma},\gimel,\varphi}(\mathbb{R} \times Q, \overline{\mathbf{F}})$ as those $\tilde{w}$ for which

- $(-\mathrm{av}_{\gimel_i}(\tilde{w})) * \tilde{w}(z) \in \widetilde{H}_{[z]}$
- The above shifted map transversally intersects $\widetilde{H}_{[z]}$

Toy Groupoidal Categories



Toy Groupoidal Categories

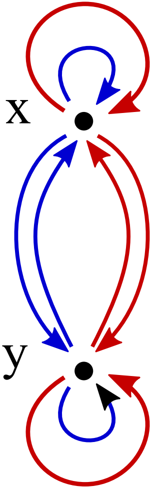


Without adding objects,

1) add the fewest morphisms to make this a groupoidal category

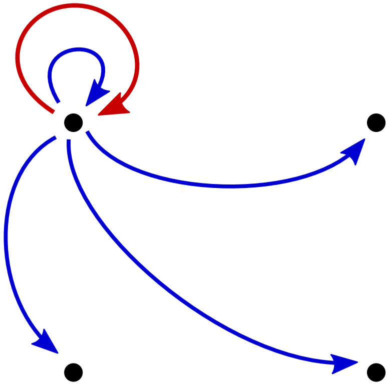
2) and without increasing the isotropy at x, add the most morphisms while keeping it a groupoidal category

Toy Groupoidal Categories



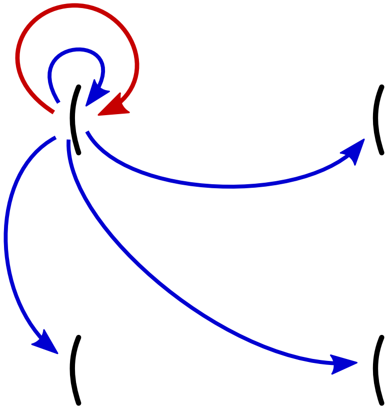
Answer

Toy Groupoidal Categories



Same questions:

Toy Groupoidal Categories



Same questions:

Definition -- Groupoidal Category

A groupoidal category is a category $\mathcal{C}$ is a category with the following properties.

1. Every morphism is an isomorphism (i.e. has an inverse).
2. Between any two objects there are only finitely many morphisms.
3. The orbit space $|\mathcal{C}|$, (collection of isomorphism classes) is a set

Definition -- Translation Groupoid

Let $\mathcal{O}$ be an M-polyfold, and G a finite group acting on $\mathcal{O}$ by sc-diffeomorphisms. Then the associated translation groupoid $G \ltimes O$ is the category with

1. Objects O
2. Morphisms $G \times O$ understood as

$$g \xrightarrow{(g,o)} g * o$$

Definition -- GCT

A GCT is a pair $(\mathcal{C}, \mathcal{T})$ where $\mathcal{C}$ is a groupoidal category and $\mathcal{T}$ is a metrizable topology on the orbit space $|\mathcal{C}|$.

Definition -- Uniformizer

Given groupoidal category $\mathcal{C}$, a uniformizer at $c \in \mathrm{Ob}(\mathcal{C})$ with automorphism group G, is a functor $\Psi : G \ltimes O \to \mathcal{C}$ with the following properties.

1. O is an M-polyfold
2. G acts on O via sc-diffeomorphism
3. $G \ltimes O$ is assoc. translation groupoid
4. there exists $\bar{o} \in O$ s.t. $\Psi(\bar{o}) = c$
5. Ψ is injective on objects
6. Ψ is full and faithful

Definition -- Uniformizer Construction

A uniformizer construction is a functor $F : \mathcal{C} \to \text{SET}$ which associates to an object c a set of uniformizers. If for each object c, the set $F(c)$ contains only tame uniformizers, then we shall call F a *tame uniformizer construction*.

Definition -- Transition Set

Fix a groupoidal category $\mathcal{C}$ and a local uniformizer construction $F : \mathcal{C} \to \mathrm{SET}$, $\alpha, \alpha' \in \mathrm{Ob}(\mathcal{C})$, and local uniformizers $\Psi \in F(\alpha)$ and $\Psi' \in F(\alpha')$, so that

$$G \ltimes O \xrightarrow{\Psi} \mathcal{C} \xleftarrow{\Psi'} G' \ltimes O'$$

Define the transition set $\mathbf{M}(\Psi, \Psi')$ by

$$\mathbf{M}(\Psi, \Psi') = \big\{(o, \Phi, o') : o \in O,\ o' \in O', \\ \Phi \in \mathrm{Hom}\big(\Psi(o), \Psi'(o')\big)\big\}$$

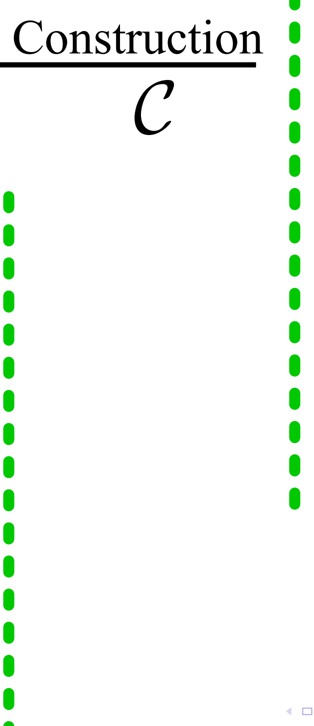
Definition -- Transition Set

$$\mathbf{M}(\Psi,\Psi') = \Big\{(o,\Phi,o') : o \in O,\ o' \in O',\\ \Phi \in \mathrm{Hom}\big(\Psi(o),\Psi'(o')\big)\Big\}$$

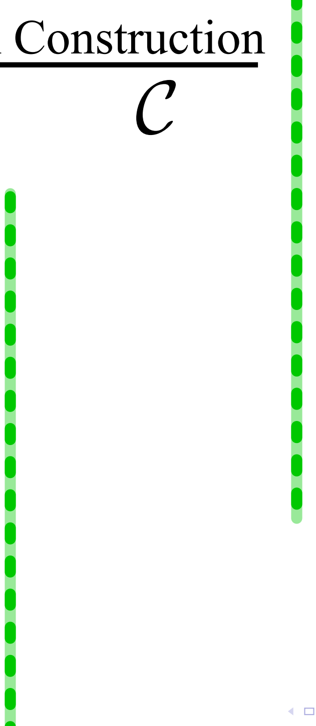
Recall that the transition set $\mathbf{M}(\Psi,\Psi')$ is equipped with the following structure maps.

1. **source map**
2. **target map**
3. **unit map** (identity)
4. **inversion map**
5. **multiplication map** (composition)

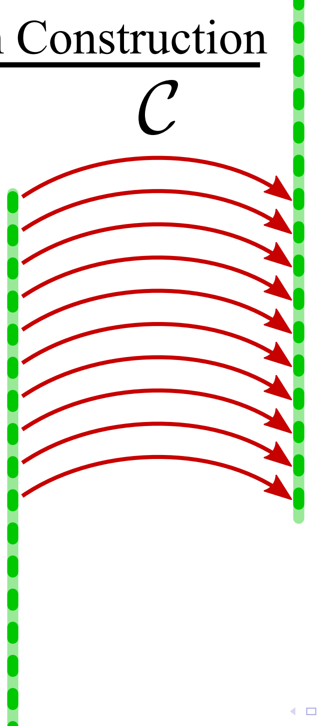
Transition Germ Construction



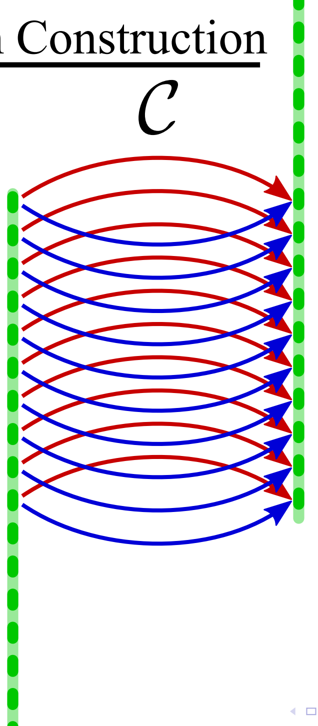
Transition Germ Construction



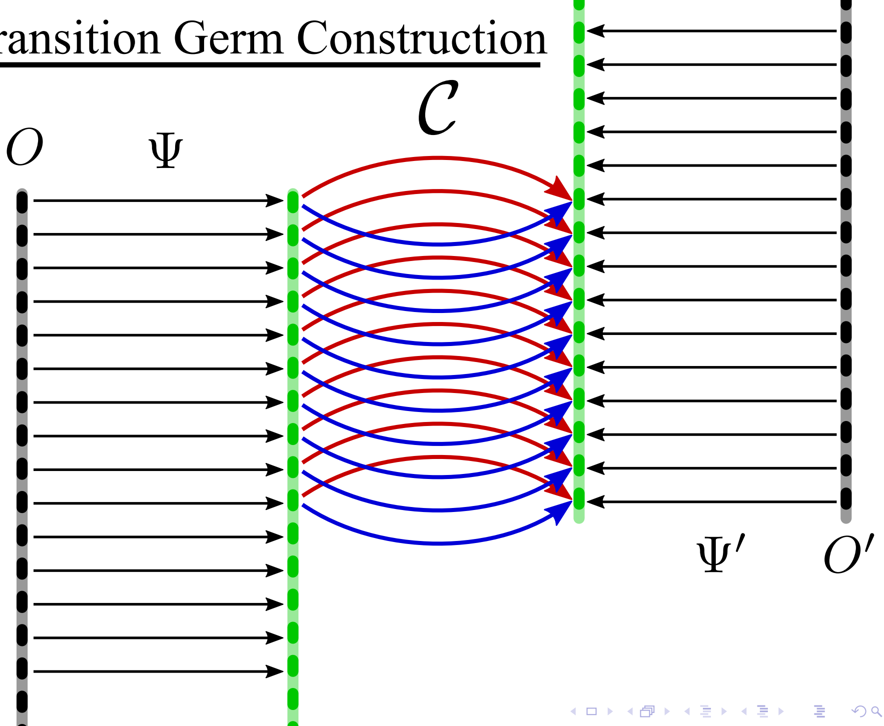
Transition Germ Construction



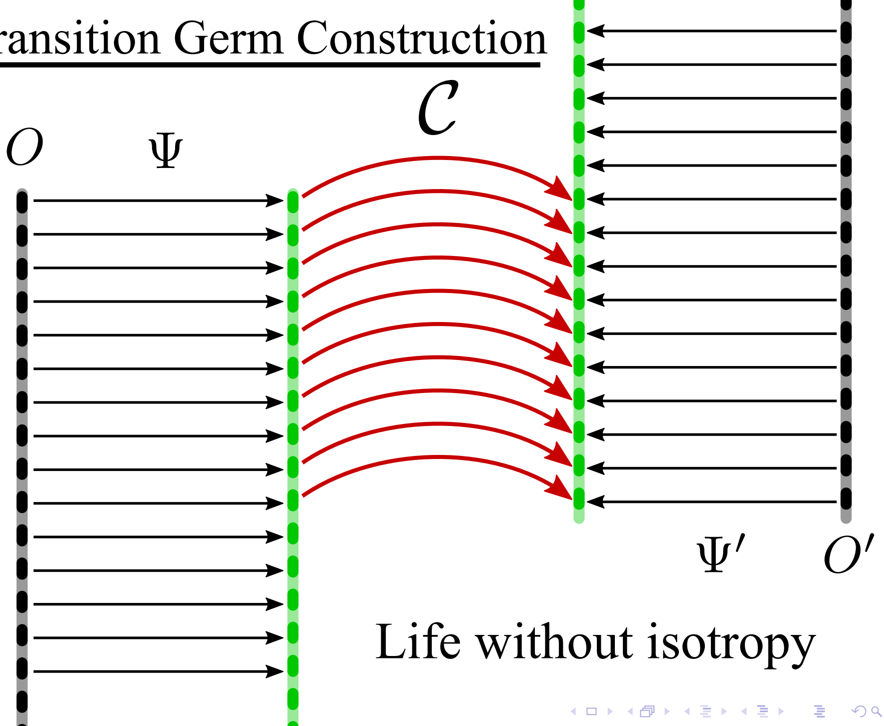
Transition Germ Construction



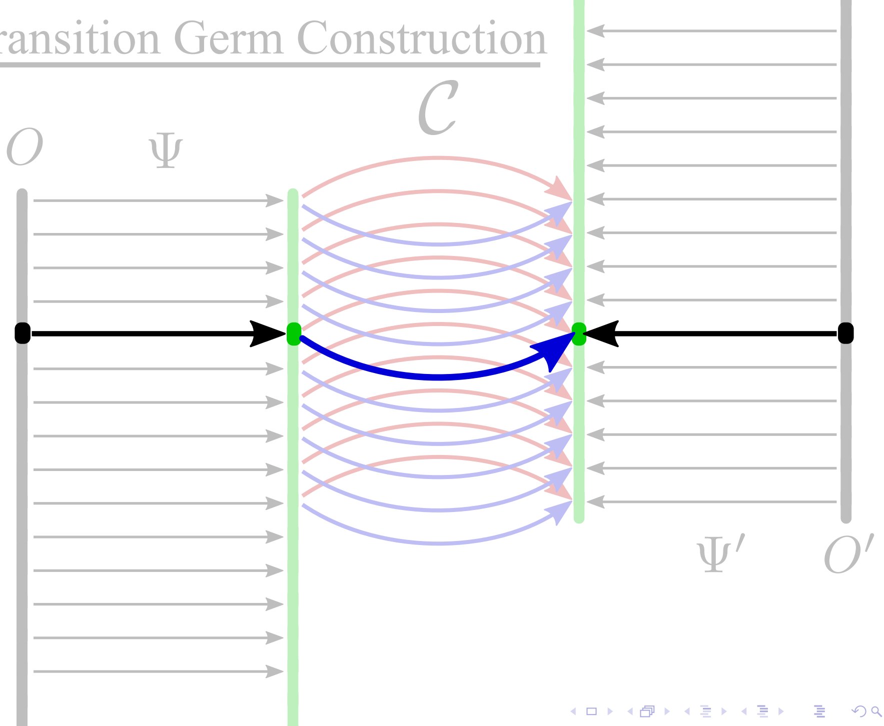
Transition Germ Construction



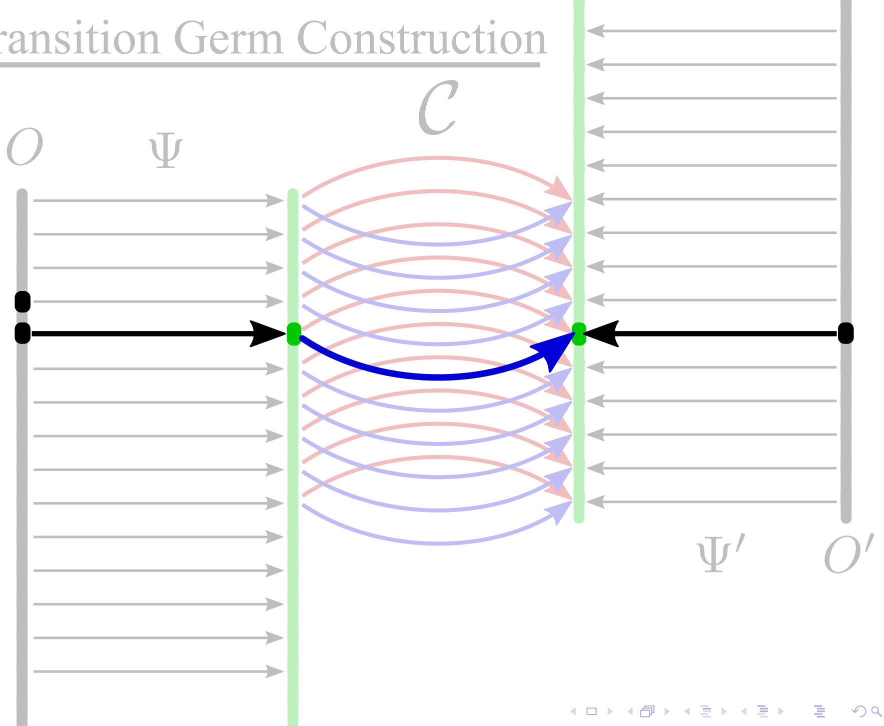
Transition Germ Construction



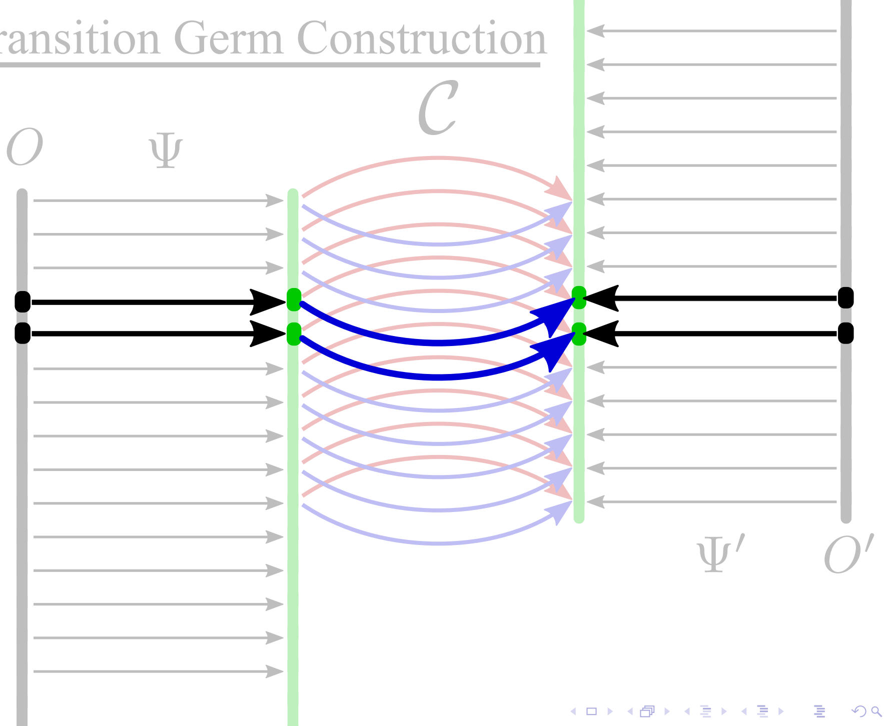
Transition Germ Construction



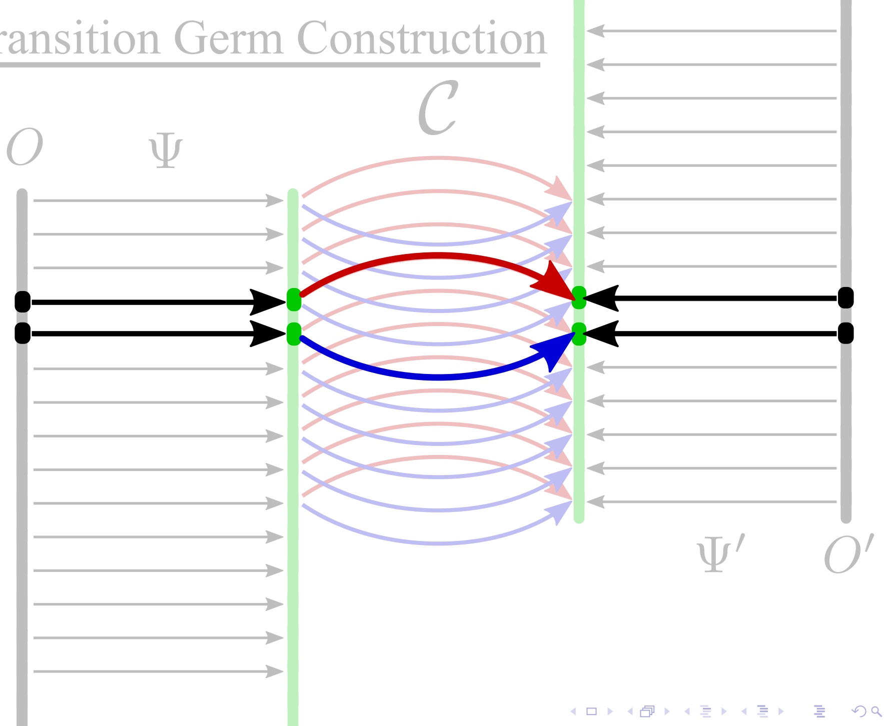
Transition Germ Construction



Transition Germ Construction



Transition Germ Construction



Transition Germ Construction

Let F be a uniformizer construction. A transition germ construction $\mathcal{G}$ associates for given $\Psi \in F(c)$ and $\Psi \in F(c')$ to $h = (o, \Phi, o') \in \mathbf{M}(\Psi, \Psi')$ a germ of map $\mathfrak{G}_h : (\mathcal{O}, o) \to (\mathbf{M}(\Psi, \Psi'), h)$ with the following properties, where $\mathfrak{g}_h = t \circ \mathfrak{G}_h$.

Transition Germ Construction

Diffeomorphism Property:
The germ $\mathfrak{g}_h : \mathcal{O}(O, o) \to \mathcal{O}(O', o')$ is a local sc-diffeomorphism and $s(\mathfrak{G}_h(q)) = q$ for q near o. If $\Psi = \Psi'$ and $h = (o, \Psi(g, o), g * o)$ then $\mathfrak{G}_h(q) = (q, \Psi(g, q), g * q)$ for q near o so that $f_h(q) = g * q$.

Transition Germ Construction

Stability Property:

$\mathfrak{G}_{\mathfrak{G}_h(q)}(p) = \mathfrak{G}_h(p)$ for q near $o = s(h)$ and p near q.

Transition Germ Construction

Identity Property:

$\mathfrak{G}_{u(o)}(q) = u(q)$ for q near o.

Transition Germ Construction

Inversion Property:

$\mathfrak{G}_{\iota(h)}(\mathfrak{g}_h(q)) = \iota(\mathfrak{G}_h(q))$ for q near $o = s(h)$.

Here $\iota(p, \Phi, o')) = (o', \Phi^{-1}, o)$.

Transition Germ Construction

Multiplication Property:

If $s(h') = t(h)$ then $\mathfrak{g}_{h'} \circ \mathfrak{g}_h(q) = \mathfrak{g}_{m(h',h)}(q)$ for q near $o = s(h)$, and $m(\mathfrak{G}_{h'}(\mathfrak{g}_h(q)), \mathfrak{G}_h(q)) = \mathfrak{G}_{m(h,h')}(q)$ for q near $o = s(h)$.

Transition Germ Construction

M-Hausdorff Property:
For different $h_1, h_2 \in \mathbf{M}(\Psi, \Psi')$ with $o = s(h_1) = s(h_2)$ the images under $\mathfrak{G}_{h_1}$ and $\mathfrak{G}_{h_2}$ of small neighborhoods are disjoint.

Upshot:

Key upshot of transition germ construction:

1. Natural topology $\mathcal{T}$ on $|\mathcal{C}|$
2. $|\Psi| : |O| \to |\mathcal{C}|$ are homeomoprhisms with image
3. induces M-polyfold structures on the $\mathbf{M}(\Psi, \Psi')$.

Moreover:

4. If $\mathcal{T}$ is metrizable, then $(\mathcal{C}, \mathcal{T})$ is a GCT.

(this is the case for the category of stable maps)

"Transition Category"

Objects:

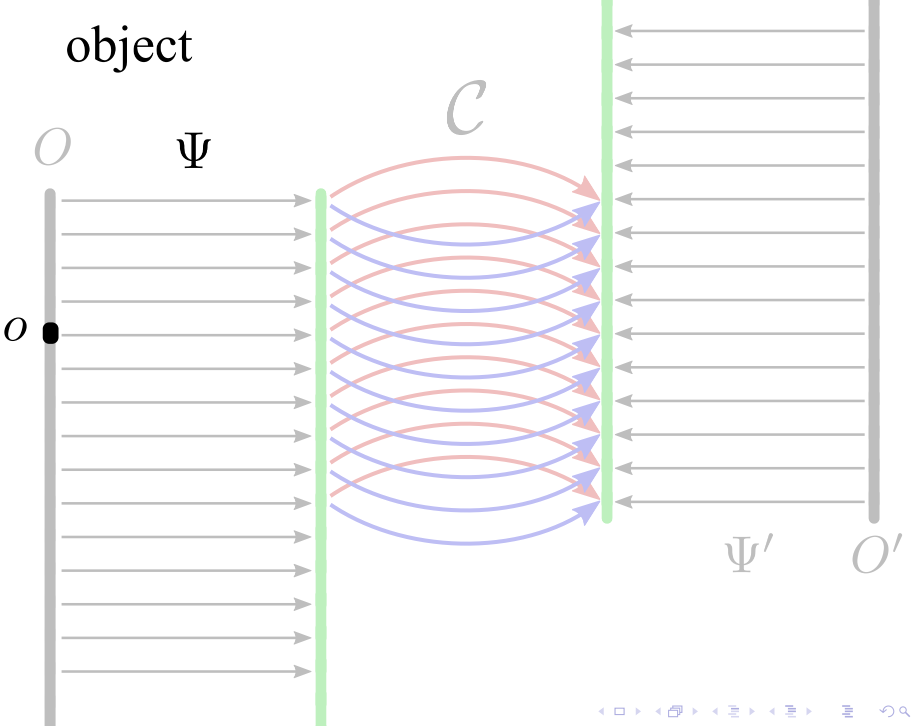
(Ψ, o) such that $\Psi : G \ltimes O \to \mathcal{C}$
$o \in O$

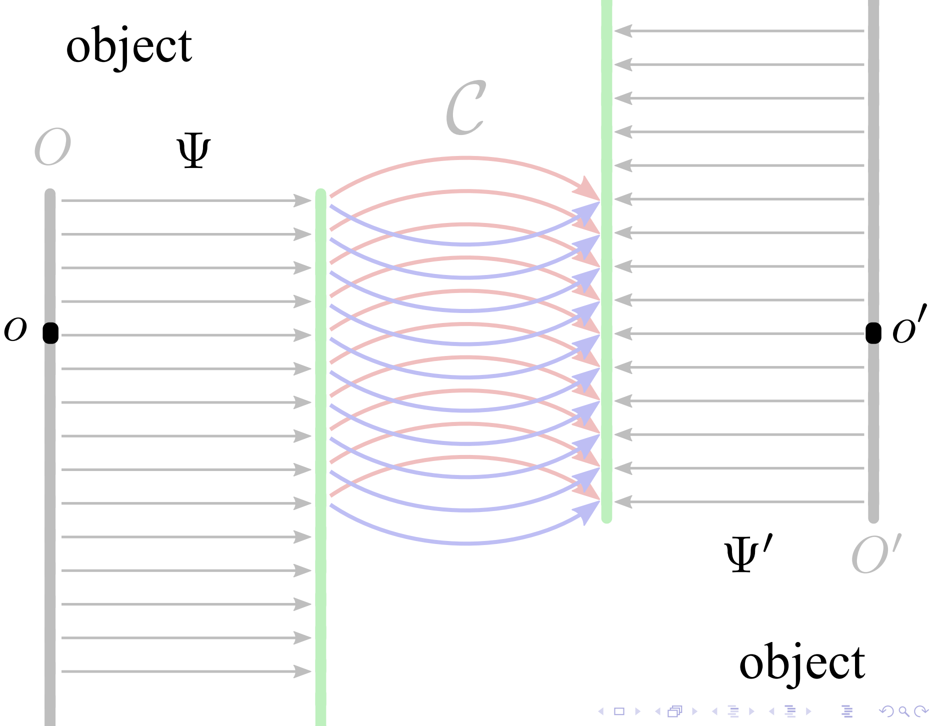
(Ψ', o') $\Psi' : G' \ltimes O' \to \mathcal{C}$,
$o' \in O'$

Morphisms:

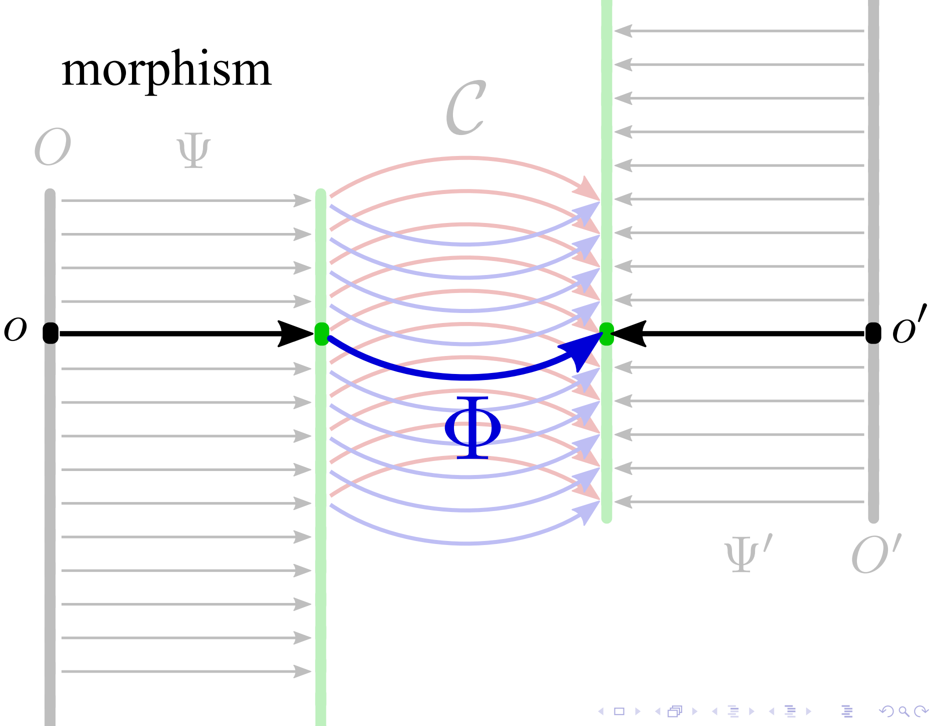
(o, Φ, o') such that $\Phi \in \mathrm{Mor}(\mathcal{C})$
$\Psi(o) \xrightarrow{\Phi} \Psi'(o')$

object

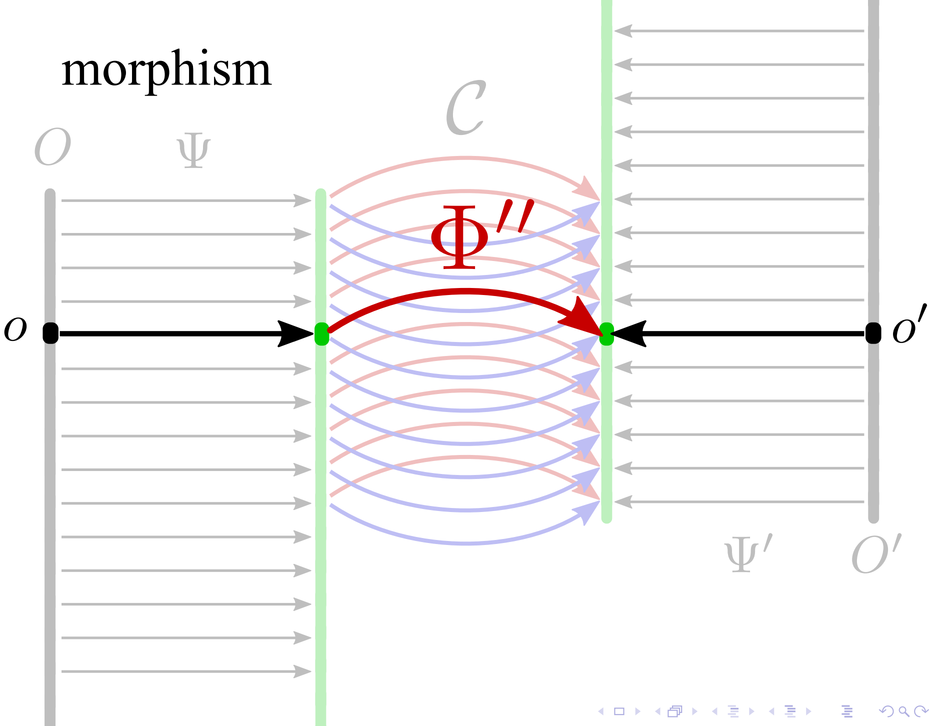



object
$\mathcal{C}$
O
Ψ
o
o'
Ψ'
O'
object

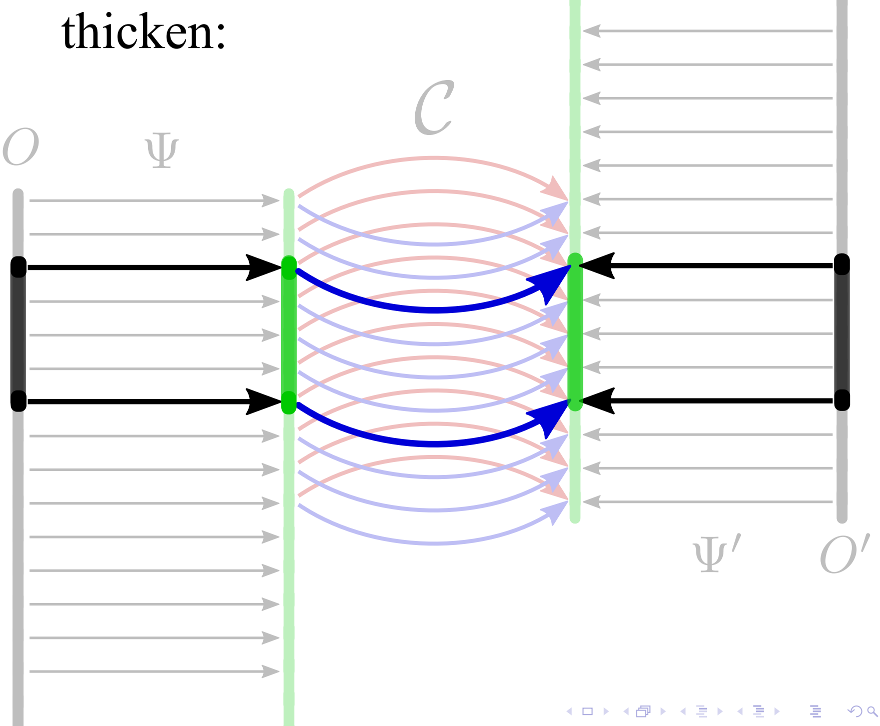
morphism



morphism



thicken:



$$\left(\Psi, o\right) \xrightarrow{\left(o,\, \Phi,\, o'\right)} \left(\Psi', o'\right)$$

"thicken"

$$\left(\Psi, \mathcal{O}(o)\right) \xrightarrow{\left(\mathcal{O}(o), \mathcal{O}(\Phi), \mathcal{O}(o')\right)} \left(\Psi', \mathcal{O}(o')\right)$$

Building charts/uniformizers:

Given

(Q, λ, ω)

J

$\delta_J : \mathcal{P}^* \to (0, 2\pi]$

φ

β

Base-point

α

Choose

$\mathbf{D}$

$\gimel$

$\mathcal{H}$

Ξ

Determined

$\mathcal{S}^{3,\delta_0}(Q, \lambda, \omega)$

σ $\bar{\sigma}$

G G^*

Review: Collecting Pieces

Given (background structures)

1. (Q, λ, ω)
   (a) Q closed odd dimensional manifold
   (b) (λ, ω) non-degenerate stable Hamiltonian structure
2. compatible/admissible almost complex structure J
3. *determine* spectral gap map, $\delta_J : \mathcal{P}^* \to (0, 2\pi]$
4. *choose* associated weight sequences $\boldsymbol{\gamma} \mapsto \bar{\boldsymbol{\gamma}}$
5. *define* category of stable maps $\mathcal{S}^{3,\delta_0}(Q, \lambda, \omega)$

Review: Collecting Pieces

Choices (for charts)

1. $\alpha = (\alpha_0, \hat{b}_1, \ldots, \hat{b}_k, \alpha_k)$ with isotropy group G
2. *determines* underlying $\sigma = (\sigma_0, b_1, \ldots, b_k, \sigma_k)$
3. *choose* stabilization set Ξ with associated transversal constraints $\mathcal{H}_{[z]}$ (two types)
4. *choose* small disk structure $\mathbf{D}$ and anchor points Υ
5. verify that ...(see next slide)

Review: Collecting Pieces

5. verify that

- the sets $M, \Gamma, ⅃, \Xi, D$ are all pairwise disjoint
- **D** is disjoint from $M, ⅃, \Xi$
- the sets $M, \Gamma, ⅃, \Xi, D$ and **D** are G-invariant
- the Riemann surface $\bar{\sigma} = (S, j, \overline{M}, \overline{D})$ is stable where

$$\overline{M} = M \cup \Gamma_0^- \cup \Gamma_k^+ \cup \Xi$$

$$\overline{D} = D \cup \Big\{ \{z, b_i(z)\} : z \in \Gamma_{i-1}^+ \;\; i \in \{1, \ldots, k\} \Big\}$$