

Workshop on Symplectic Field Theory IX: POLYFOLDS FOR SFT

Lectures 5 - 9 (version 3.0)

University of Augsburg

27 - 31 August 2018

Preamble:

Please direct corrections, comments, etc. to
`joel.fish@umb.edu`

Work in progress:
`www.polyfolds.org`

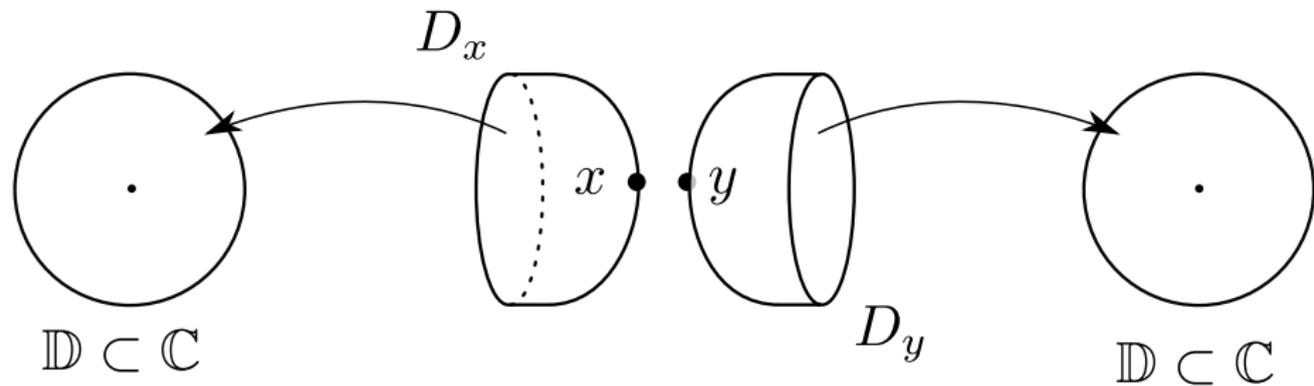
Hopeful idea: Polyfold Summer School

Topics:

1. Toy-model M-polyfold (standard node)
2. Imprinting method (theory & example)
3. Imprinting plus operations
4. A basic "LEGO" block
5. New blocks from old (theory & example)
6. Periodic orbits and nodal interface pairs
7. Preliminary "LEGO" building

Nodal Disk Pair

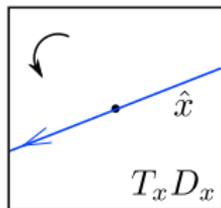
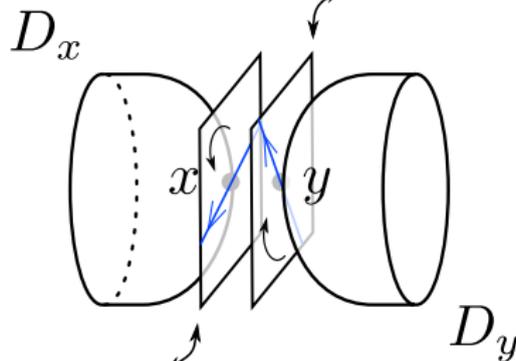
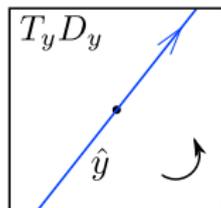
$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$



Nodal Disk Pair

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

$\hat{y} \subset T_y D_y$ is an oriented real line

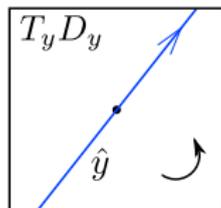


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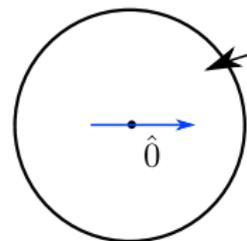
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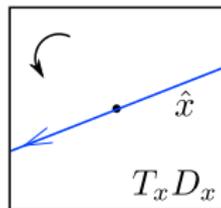
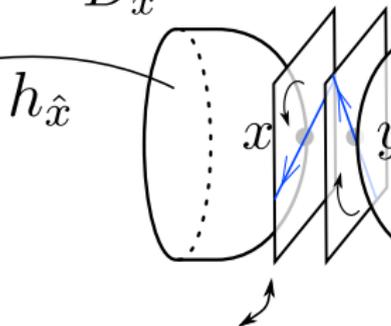
$$h_{\hat{x}}(x) = 0$$

$$Th_{\hat{x}}(\hat{x}) = \hat{0}$$



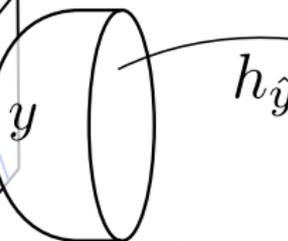
$$\mathbb{D} \subset \mathbb{C}$$

D_x



$$h_{\hat{y}}(y) = 0$$

$$Th_{\hat{y}}(\hat{y}) = \hat{0}$$



D_y

$$\mathbb{D} \subset \mathbb{C}$$

$\hat{x} \subset T_x D_x$ is an oriented real line

Natural angles

Circle action on decorations:

$$(\theta, \hat{x}) \rightarrow \theta * \hat{x} := e^{2\pi i \theta} \hat{x}$$

Equivalence relation on decorated nodal pairs:

$$\{\hat{x}, \hat{y}\} \sim \{\hat{x}', \hat{y}'\} \text{ iff } \exists \theta \in S^1 = \mathbb{R}/\mathbb{Z}$$

such that

$$\hat{x}' = \theta * \hat{x} \text{ and } \hat{y}' = \theta^{-1} * \hat{y}$$

Natural angles

Circle action on decorations:

$$(\theta, \hat{x}) \rightarrow \theta * \hat{x} := e^{2\pi i \theta} \hat{x}$$

A natural angle is then defined as an element in the associated equivalence class, or alternatively as

$$[\hat{x}, \hat{y}] = \{ \{ \theta * \hat{x}, \theta^{-1} * \hat{y} \} : \theta \in S^1 \}$$

Gluing Parameters

Associated to a nodal disk pair

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

we define the associated set of gluing parameters

$$\mathbb{B}_{\mathcal{D}}$$

as formal expressions of the form

$$r \cdot [\hat{x}, \hat{y}]$$

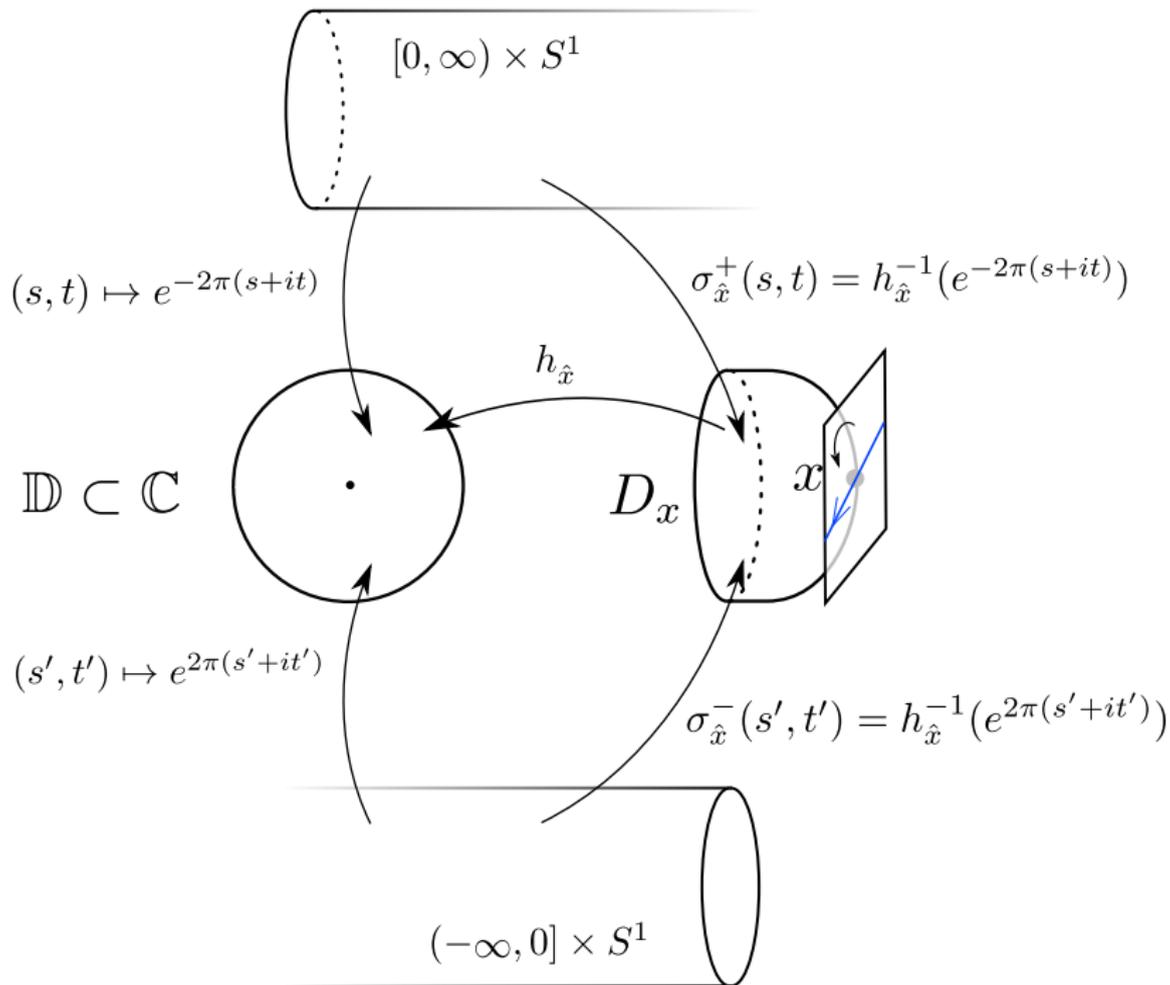
Cylinders Z_a

Given a nodal disk pair $\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$ and a gluing parameter $a = r \cdot [\hat{x}, \hat{y}] \in \mathbb{B}_{\mathcal{D}}$ with $r > 0$ define the cylinder

$$Z_a = \left\{ \{z, z'\} : z \in D_x, z' \in D_y, \right. \\ \left. h_{\hat{x}}(z) \cdot h_{\hat{y}}(z') = e^{-2\pi\varphi(r)} \right\}$$

for $a = 0$ i.e. $r = 0$ define

$$Z_a = D_x \sqcup D_y$$



Cylinders Z_α

The maps

$$\sigma_{\hat{x}}^+ : [0, \infty) \times S^1 \rightarrow D_x$$

$$\sigma_{\hat{y}}^- : (-\infty, 0] \times S^1 \rightarrow D_y$$

induces coordinates on D_x and D_y via

$$z = (s, t) \in [0, \infty) \times S^1 \quad \text{for } z \in D_x$$

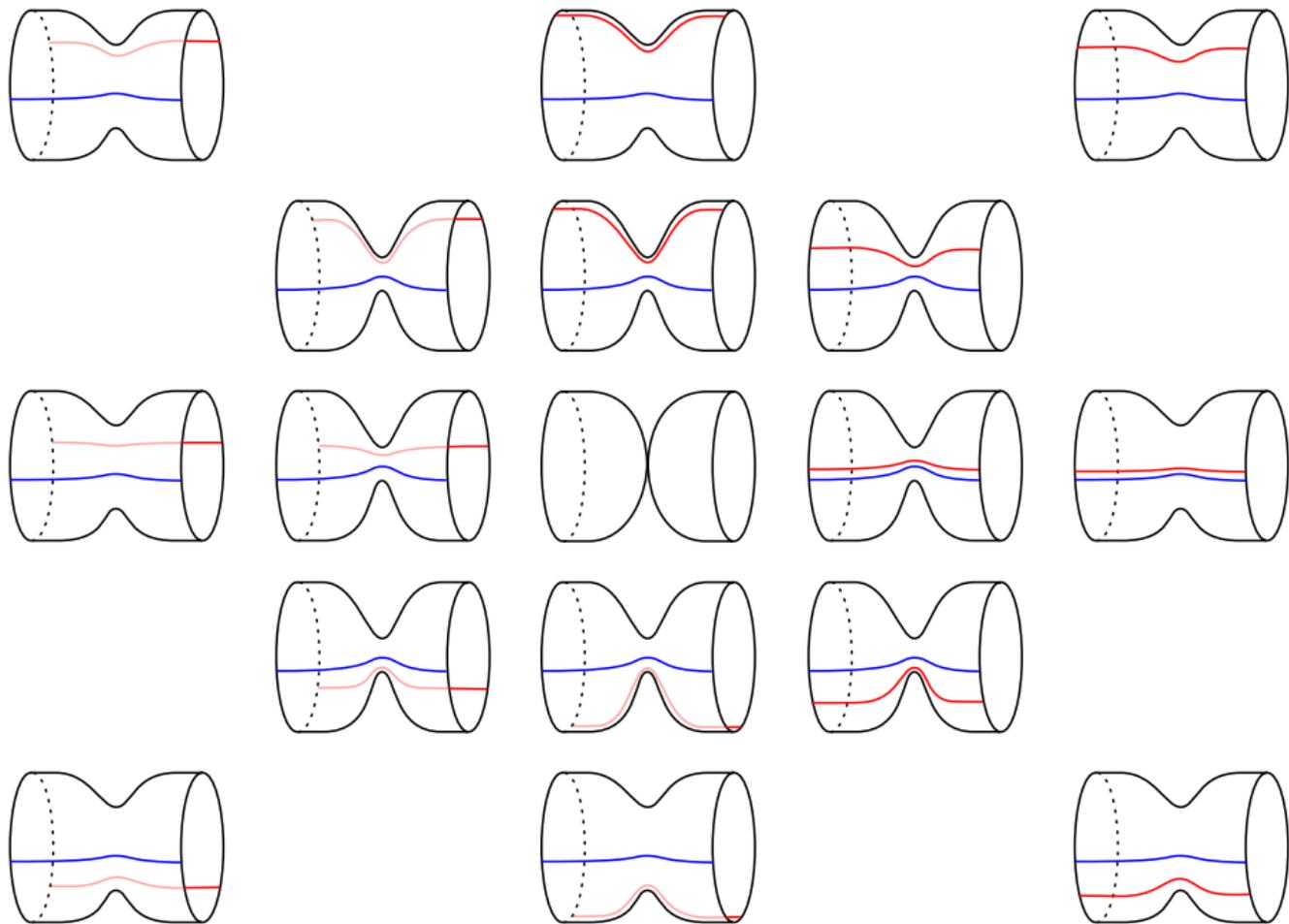
$$z' = (s', t') \in (-\infty, 0] \times S^1 \quad \text{for } z' \in D_y$$

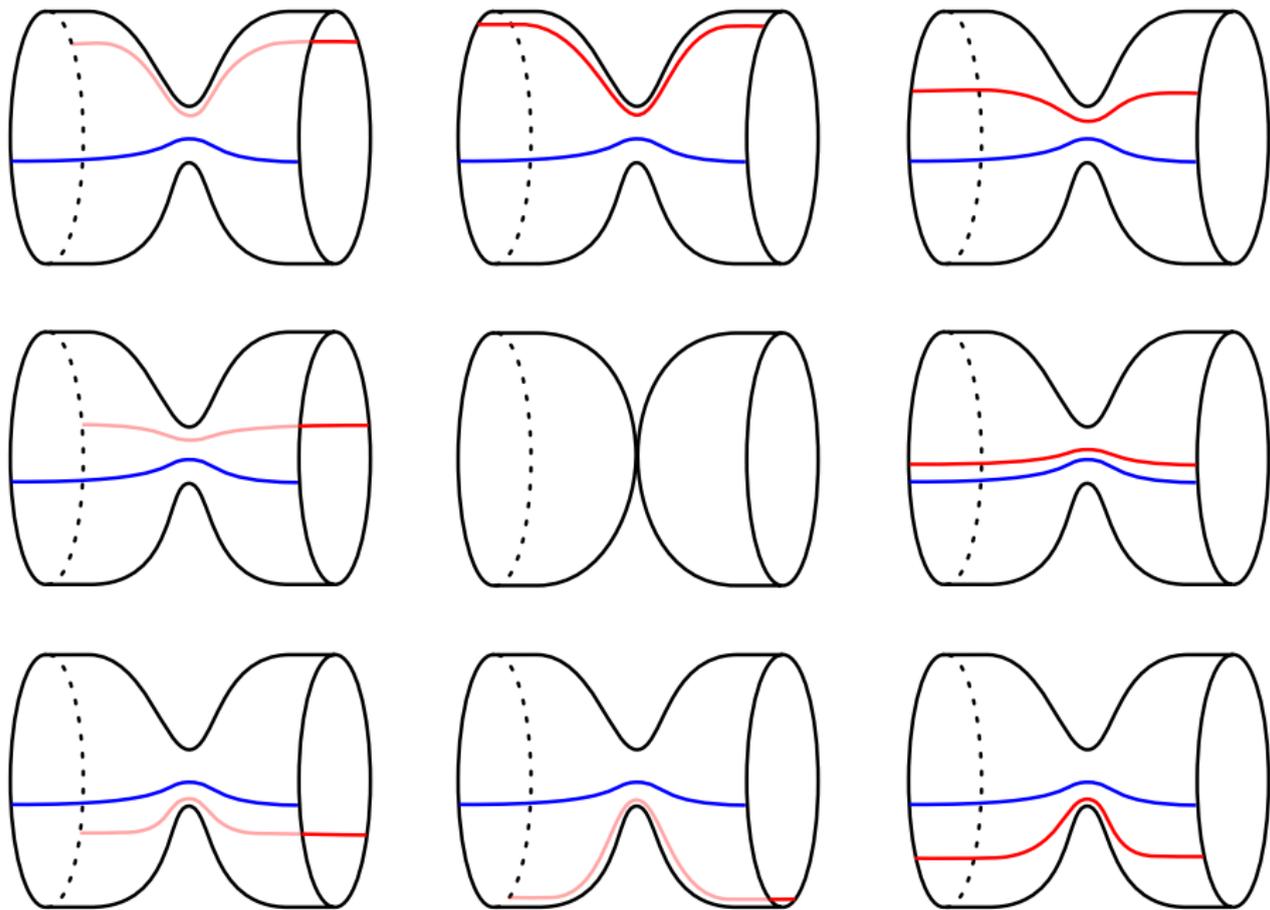
Cylinders Z_a

These induce coordinates on the Z_a which can alternately be described as

$$Z_a = \left\{ \left\{ (s, t), (s', t') \right\} : \begin{aligned} (s, t) &\in [0, R] \times S^1, \\ (s', t') &\in [-R, 0] \times S^1, \\ s &= s' + R, \\ t &= t' + \theta \end{aligned} \right\}$$

where $R = \varphi(|a|)$





Cylinders Z_a Takeaway

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\}) \rightsquigarrow \mathbb{B}_{\mathcal{D}}$$

$$(\mathbb{B}_{\mathcal{D}}, \mathcal{D}) \rightsquigarrow \bigcup_{a \in \mathbb{B}_{\mathcal{D}}} Z_a$$

$$a \neq a' \implies Z_a \neq Z_{a'}$$

$$a = r \cdot [\hat{x}, \hat{y}] \in \mathbb{B}_{\mathcal{D}}$$

Disconnected Function Spaces

$$\delta : 0 < \delta_0 < \delta_1 < \dots$$

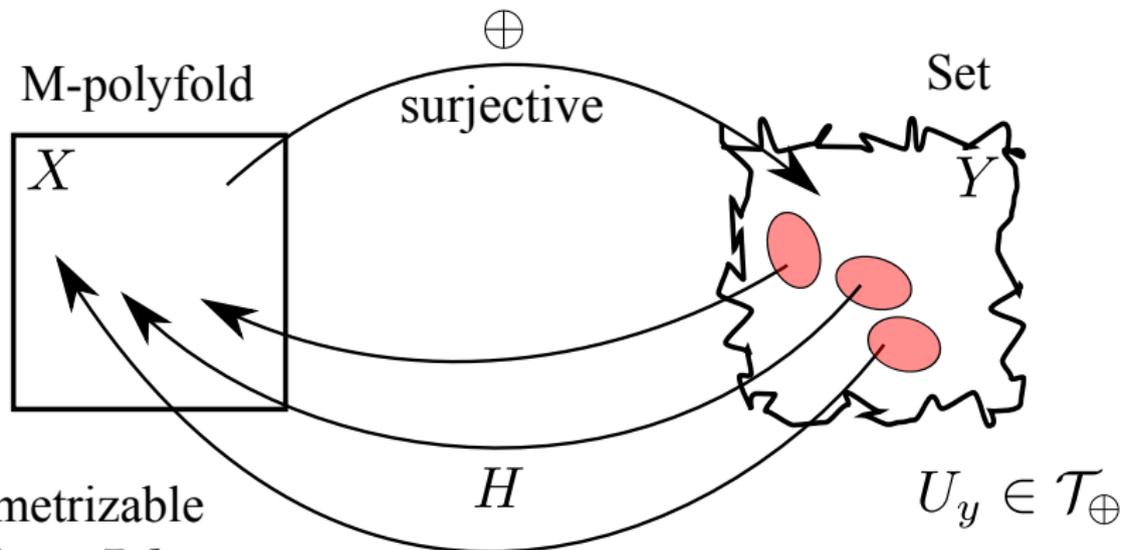
$$E_{\mathcal{D}}^{\delta_0} = \mathbb{R}^N \oplus H^{3,\delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

$$X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) = E_{\mathcal{D}}^{\delta_0} \sqcup \left(\bigcup_{0 < |a| < \frac{1}{4}} H^3(Z_a, \mathbb{R}^N) \right)$$

We aim to equip $X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)$ with an M-polyfold structure

Theorem: Imprinting Method

Given:



\mathcal{T}_\oplus is metrizable

$$\oplus \circ H = Id$$

$$H \circ \oplus \text{ is } sc^\infty$$

Then:

- Y is an M-polyfold
- \oplus and each H is sc^∞

Specific Imprinting

$$\oplus : \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \rightarrow X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)$$

$$\oplus_a(u^+, u^-) : Z_a \rightarrow \mathbb{R}^N$$

$$\oplus_a(u^+, u^-)(\{(s, t), (s', t')\}) =$$

$$\beta(|s| - \frac{1}{2}R) \cdot u^+(s, t) + \beta(|s'| - \frac{1}{2}R) \cdot u^-(s', t')$$

Recall:

$$E_{\mathcal{D}}^{\delta_0} = \mathbb{R}^N \oplus H^{3,\delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

$$X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) = E_{\mathcal{D}}^{\delta_0} \sqcup \left(\bigcup_{0 < |a| < \frac{1}{4}} H^3(Z_a, \mathbb{R}^N) \right)$$

Housekeeping Theorem 1

Given:

Imprintings

$$\oplus_1 : X_1 \rightarrow Y_1$$

$$\oplus_2 : X_2 \rightarrow Y_2$$

Then:

$$\oplus_1 \times \oplus_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$$

$$\oplus_1 \sqcup \oplus_2 : X_1 \sqcup X_2 \rightarrow Y_1 \sqcup Y_2$$

are imprintings

Example: Disjoint Union

Given:

two nodal disk pairs

$$\mathcal{D}_1 = (D_{x_1} \sqcup D_{y_1}, \{x_1, y_1\})$$

$$\mathcal{D}_2 = (D_{x_2} \sqcup D_{y_2}, \{x_2, y_2\})$$

and imprintings

$$\oplus_1 : \mathbb{B}_{\mathcal{D}_1} \times E_{\mathcal{D}_1}^{\delta_0} \rightarrow X_{\mathcal{D}_1, \varphi}^{\delta_0}(\mathbb{R}^N)$$

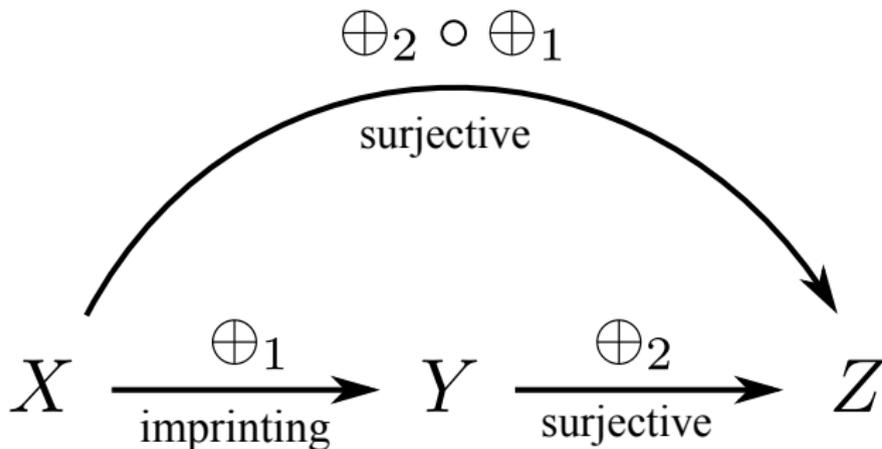
$$\oplus_2 : \mathbb{B}_{\mathcal{D}_2} \times E_{\mathcal{D}_2}^{\delta_0} \rightarrow X_{\mathcal{D}_2, \varphi}^{\delta_0}(\mathbb{R}^N)$$

Then:

- $X_{\mathcal{D}_1, \varphi}^{\delta_0}(\mathbb{R}^N) \sqcup X_{\mathcal{D}_2, \varphi}^{\delta_0}(\mathbb{R}^N)$ is an M-polyfold
- $\oplus_1 \sqcup \oplus_2$ is an imprinting

Housekeeping Theorem 2

Given:

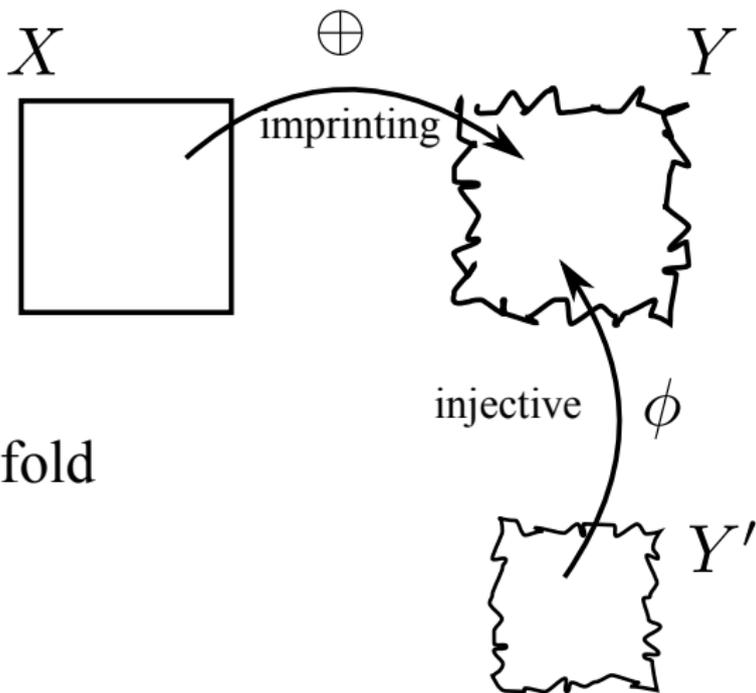


Then:

- \oplus_2 is an imprinting if and only if
- $\oplus_2 \circ \oplus_1$ is an imprinting
- Moreover: coherence.

Housekeeping Theorem 3

Given:



$X' := \oplus^{-1}(\phi(Y'))$
is a sub-M-polyfold

Then:

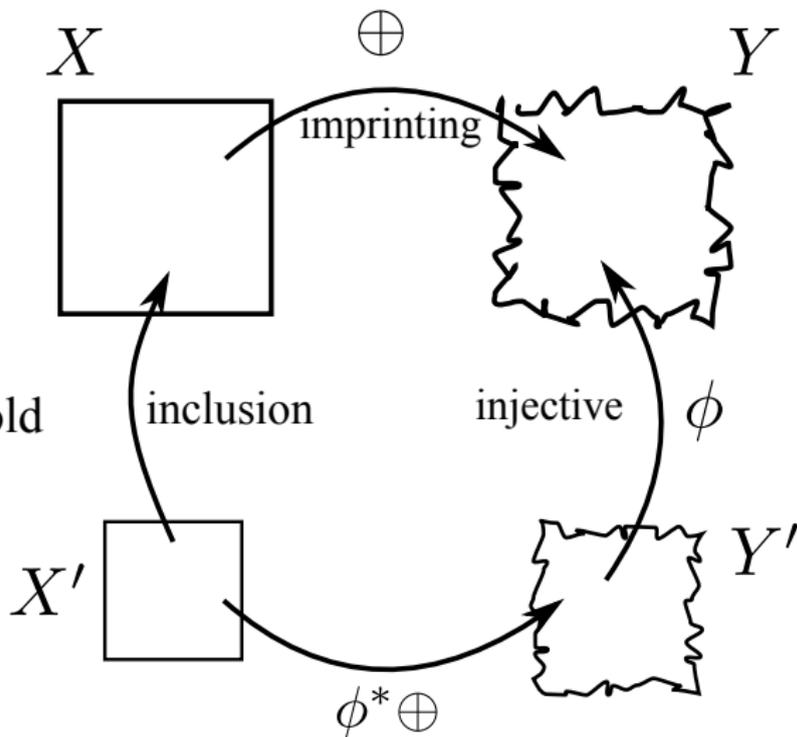
$\phi^* \oplus : X' \rightarrow Y'$ is an imprinting, where

$$\phi^* \oplus := \phi^{-1} \circ \oplus|_{X'}$$

Housekeeping Theorem 3

Given:

$X' := \oplus^{-1}(\phi(Y'))$
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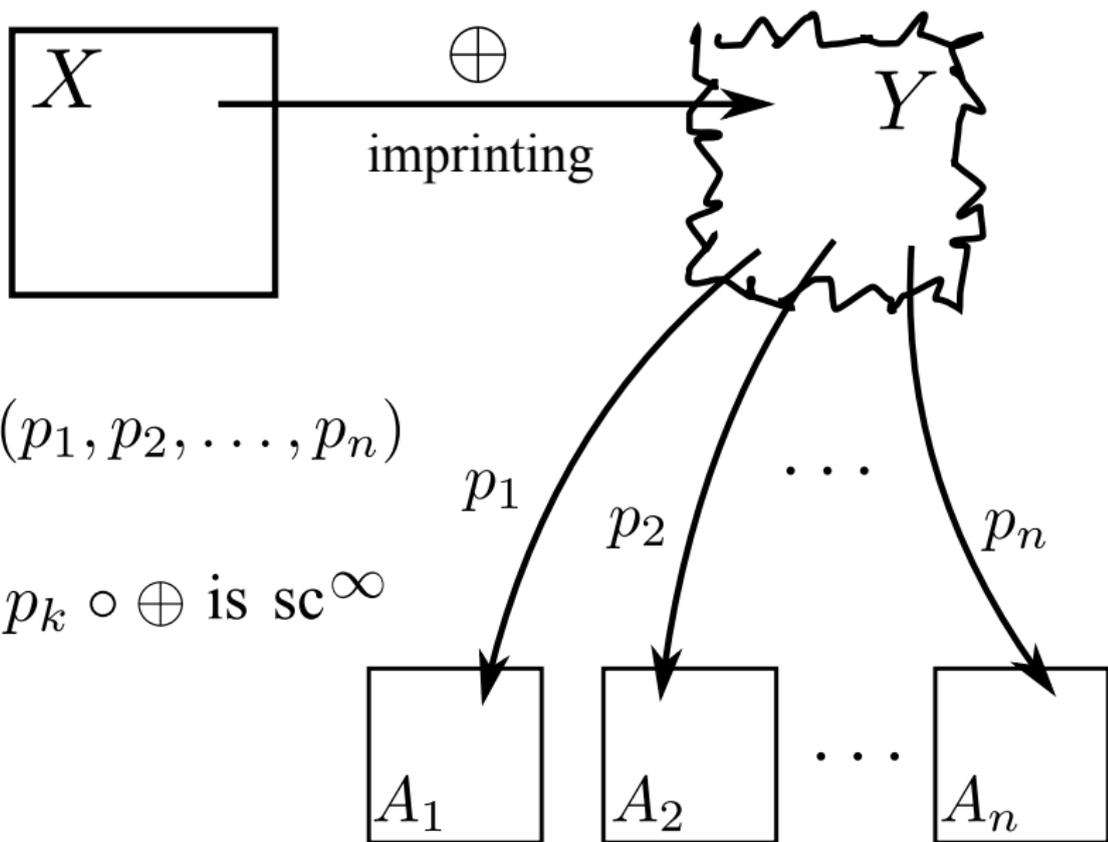


Then:

$\phi^* \oplus : X' \rightarrow Y'$ is an imprinting, where

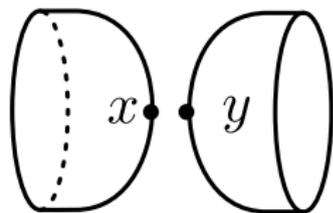
$$\phi^* \oplus := \phi^{-1} \circ \oplus|_{X'}$$

Imprinting with restrictions (\oplus, \mathbf{p})



Imprinting with restrictions -- Example

Recall, the nodal disk pair



$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

gives rise to the imprinting

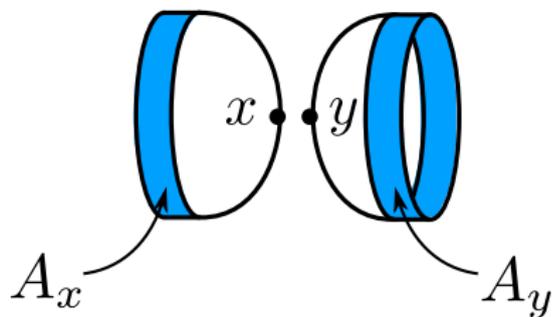
$$\oplus : \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \rightarrow X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N)$$

$$E_{\mathcal{D}}^{\delta_0} = \mathbb{R}^N \oplus H^{3, \delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

$$X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N) = E_{\mathcal{D}}^{\delta_0} \sqcup \left(\bigcup_{0 < |a| < \frac{1}{4}} H^3(Z_a, \mathbb{R}^N) \right)$$

Imprinting with restrictions -- Example

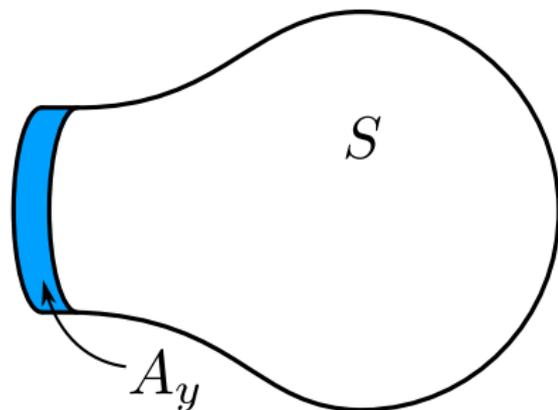
$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$



yields and imprinting with restrictions

$$\begin{array}{ccc} \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} & \xrightarrow{\oplus} & X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N) \\ & & \begin{array}{l} \swarrow p_x \\ \searrow p_y \end{array} \\ & & \begin{array}{cc} H^3(A_x, \mathbb{R}^N) & H^3(A_y, \mathbb{R}^N) \end{array} \end{array}$$

Imprinting with restrictions -- Example

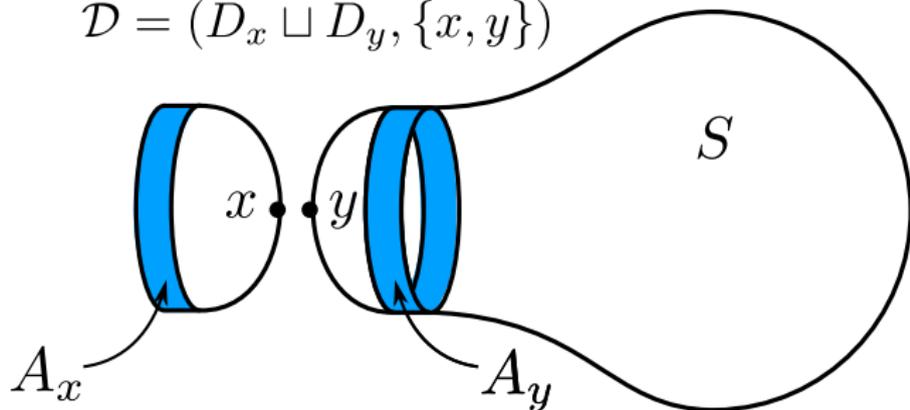


yields and imprinting with restrictions

$$\begin{array}{ccccc} \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} & \xrightarrow{\oplus} & X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N) & & H^3(S, \mathbb{R}^N) \\ & & \begin{array}{l} \swarrow p_x \\ \searrow p_y \end{array} & & \swarrow p'_y \\ & & H^3(A_x, \mathbb{R}^N) & & H^3(A_y, \mathbb{R}^N) \end{array}$$

Imprinting with restrictions -- Example

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

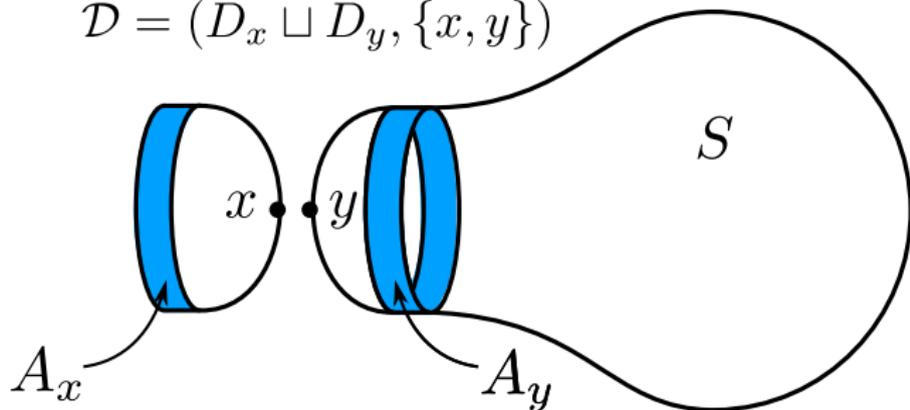


yields the M-polyfold

$$X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N)_{p_y \times p'_y} H^3(S, \mathbb{R}^N)$$

Imprinting with restrictions -- Example

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$



and more importantly yields an imprinting with restrictions

$$(\oplus \times Id)^{-1}(X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)_{p_y \times p'_y} H^3(S, \mathbb{R}^N)) \xrightarrow{\phi^*(\oplus \times Id)} X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)_{p_y \times p'_y} H^3(S, \mathbb{R}^N)$$

where ϕ is the inclusion

$$X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)_{p_y \times p'_y} H^3(S, \mathbb{R}^N) \xrightarrow{\phi} X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) \times H^3(S, \mathbb{R}^N)$$

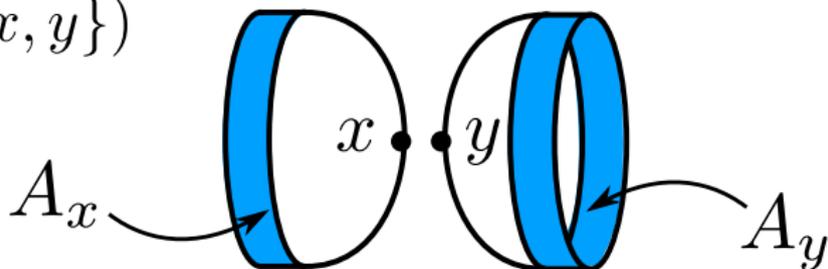
$$\begin{array}{c} \downarrow p_x \\ H^3(A_x, \mathbb{R}^N) \end{array}$$

Imprinting with restrictions -- Theorem

The fiber product over annular restrictions of imprintings with restrictions, is again an imprinting with restrictions

Feature: Projection to gluing parameter

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$



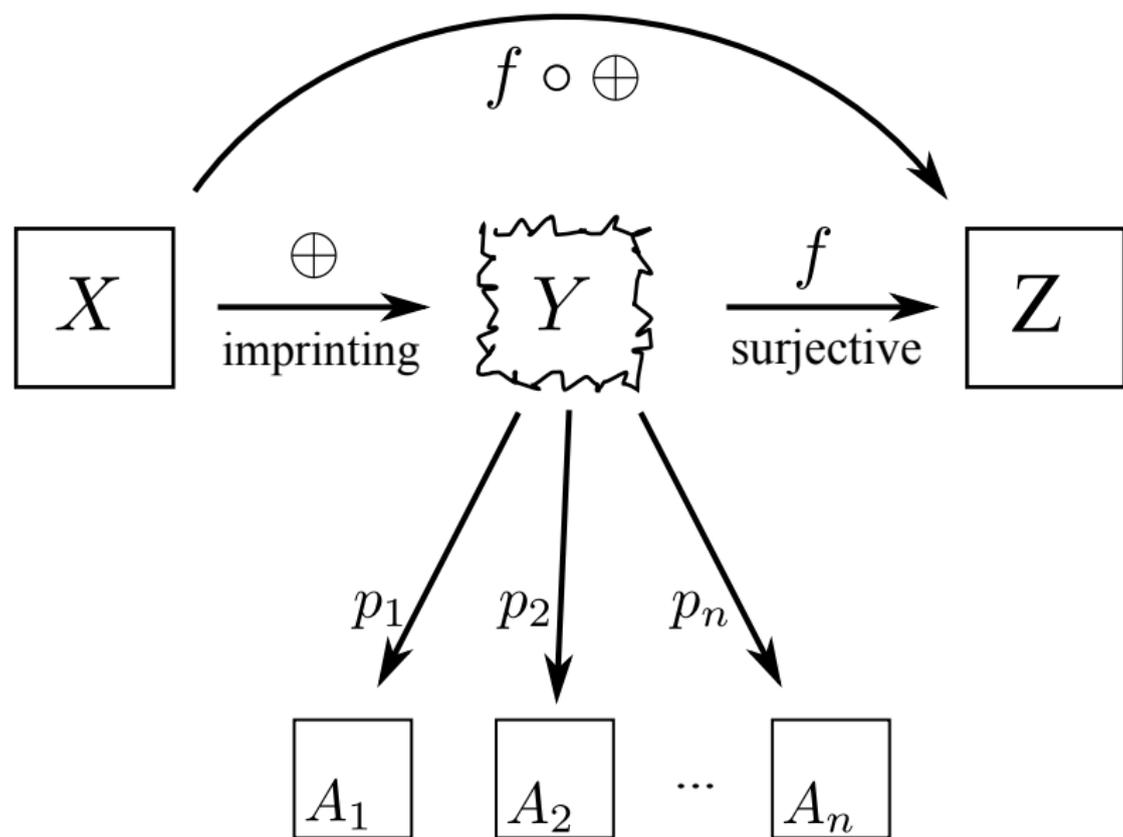
$$\begin{array}{ccccc} \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} & \xrightarrow{\oplus} & X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N) & \xrightarrow{p_{\mathbb{B}_{\mathcal{D}}}} & \mathbb{B}_{\mathcal{D}} \\ & & \swarrow p_x & & \searrow p_y \\ & & H^3(A_x, \mathbb{R}^N) & & H^3(A_y, \mathbb{R}^N) \end{array}$$

Definition: Submersive imprinting w. restrictions

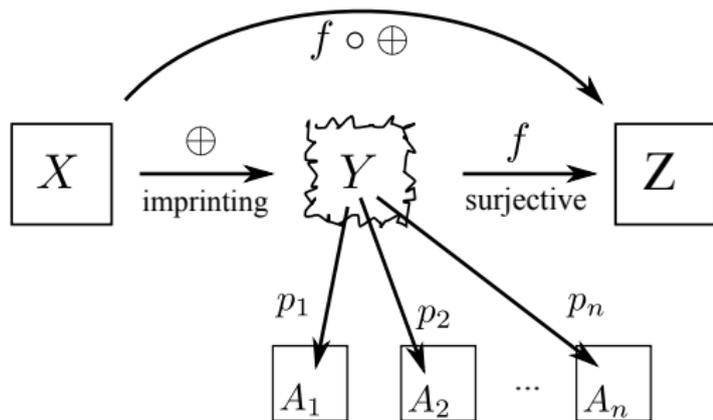
~~Definition: Submersive imprinting w. restrictions~~

Basic LEGO block

Definition: Basic LEGO Block



Definition: Basic LEGO Block



For each $(x_0, f \circ \oplus(x_0)) \in \text{Gr}(f \circ \oplus) \subset X \times Z$

there exists an open nbhd $W \subset X \times Z$ and sc-smooth map

$\rho : W \rightarrow W$ of the form $\rho(x, z) = (\bar{\rho}(x, z), z)$ such that

$$\rho \circ \rho = \rho$$

$$\rho(W) = W \cap \text{Gr}(f \circ \oplus)$$

$$p_i \circ \oplus \circ \bar{\rho}(x, z) = p_i(x)$$

Benefits of LEGO blocks:

Given LEGO blocks (\oplus, \mathbf{p}, f) and $(\oplus', \mathbf{p}', f')$
the fiber product over f and f' is another LEGO block.

If the \mathbf{p} and \mathbf{p}' are restrictions to annular neighborhoods,
then the fiber product over elements of the \mathbf{p} and \mathbf{p}'
is also another LEGO block.

From \mathbb{R}^N to manifolds

With $X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N)$ defined, we now aim to define $X_{\mathcal{D},\varphi}^{\delta_0}(Q)$ where Q is a manifold.

Let

- $\Phi : Q \rightarrow \mathbb{R}^N$ be an embedding
- $U \subset \mathbb{R}^N$ be an open neighborhood of $\Phi(Q)$
- $\text{pr} : U \rightarrow U$ a smooth retraction onto $\Phi(Q)$

$$\text{i.e.} \quad \text{pr} \circ \text{pr} = \text{pr} \quad \text{pr}(U) = \Phi(Q)$$

From \mathbb{R}^N to manifolds

Then $\mathcal{U} := \{u \in X_{\mathcal{D},\varphi}^{\delta_0}(\mathbb{R}^N) : \text{Im}(u) \subset U\}$
is open and the map

$$\begin{aligned}\rho : \mathcal{U} &\rightarrow \mathcal{U} \\ \rho(u) &= \text{pr} \circ u\end{aligned}$$

is an sc-smooth retraction.

This defines an M-polyfold structure on

$$X_{\mathcal{D},\varphi}^{\delta_0}(Q)_{\Phi,\mathbb{R}^N} = \bigcup_{a \in \mathbb{B}_{\mathcal{D}}} \{u \in \mathcal{C}^0(Z_a, Q) : \Phi \circ u \in \rho(\mathcal{U})\}$$

moreover $X_{\mathcal{D},\varphi}^{\delta_0}(Q)_{\Phi,\mathbb{R}^N} = X_{\mathcal{D},\varphi}^{\delta_0}(Q)_{\Phi',\mathbb{R}^{N'}}$ as M-polyfolds,
so we simply write $X_{\mathcal{D},\varphi}^{\delta_0}(Q)$

The periodic orbit case

Introduce

- periodic orbit: $\gamma = ([\gamma], T, k)$
- weighted periodic orbit $\bar{\gamma} = (\gamma, \delta)$
with $\delta = (\delta_k)_{k=0}^{\infty}$

- ordered nodal disk pair

$$\mathcal{D} = (D_x \sqcup D_y, (x, y))$$

The periodic orbit case

We define the function space $Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$ to be the set of tuples $(\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)$ where

$$\tilde{u}^x : D_x \setminus \{x\} \rightarrow \mathbb{R} \times \mathbb{R}^N$$

$$\tilde{u}^y : D_y \setminus \{y\} \rightarrow \mathbb{R} \times \mathbb{R}^N$$

$[\hat{x}, \hat{y}]$ is a natural angle

and for holomorphic polar coordinates $\sigma_{\hat{x}}^+$ and $\sigma_{\hat{y}}^-$ associated to a representative (\hat{x}, \hat{y}) of $[\hat{x}, \hat{y}]$ there exists $\gamma \in [\gamma]$ such that

$$\tilde{u}^x \circ \sigma_{\hat{x}}^+(s, t) = (Ts + c^x, \gamma(kt)) + \tilde{r}^x(s, t)$$

$$\tilde{u}^y \circ \sigma_{\hat{y}}^-(s', t') = (Ts' + c^y, \gamma(kt')) + \tilde{r}^y(s', t')$$

here $\tilde{r}^x, \tilde{r}^y \in H^{3, \delta_0}$

The periodic orbit case

Theorem:

$Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$ is an ssc-Hilbert manifold.

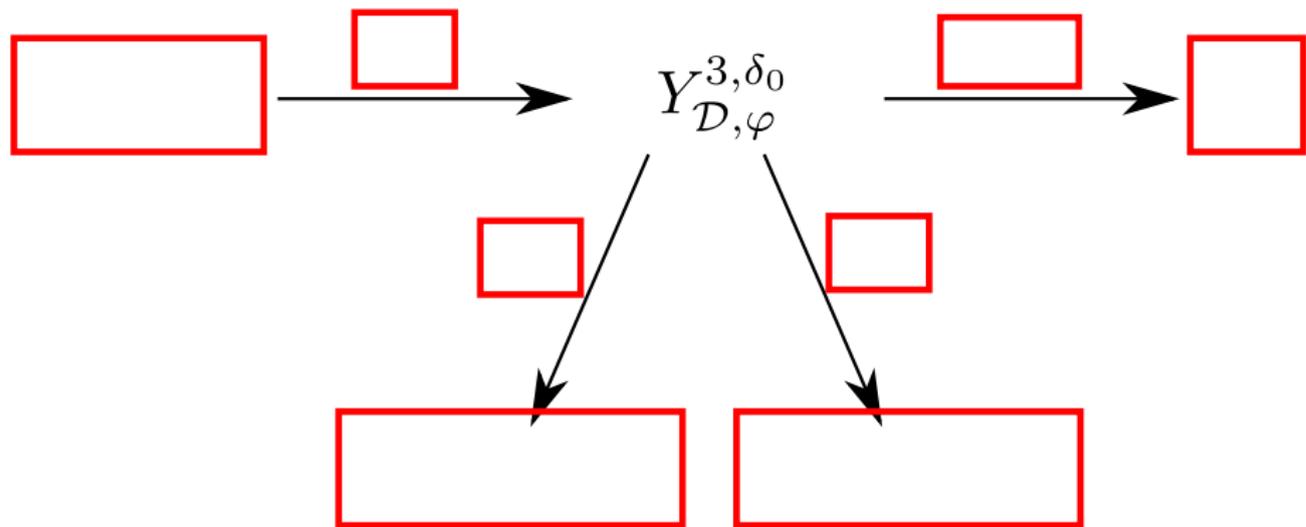
The periodic orbit case

$$\mathbb{B}_D \times E_D^{\delta_0} \xrightarrow{\oplus} X_{D,\varphi}^{\delta_0}(\mathbb{R}^N) \xrightarrow{p_{\mathbb{B}_D}} \mathbb{B}_D$$

Recall:

$$\begin{array}{ccc} & & \\ & p_x & p_y \\ & \swarrow & \searrow \\ H^3(A_x, \mathbb{R}^N) & & H^3(A_y, \mathbb{R}^N) \end{array}$$

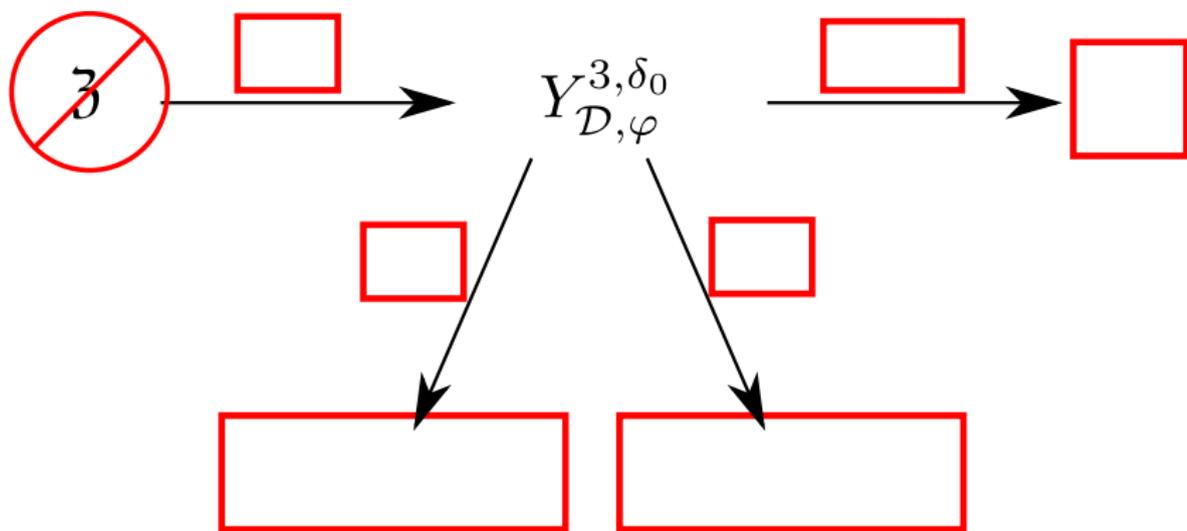
The periodic orbit case



$$Y_{\mathcal{D}, \varphi}^{3, \delta_0} = (\{0\} \times Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})) \sqcup$$

$$((0, 1) \times \bigsqcup_{0 < |a| < \frac{1}{4}} H^3(Z_a, \mathbb{R} \times \mathbb{R}^N))$$

The periodic orbit case

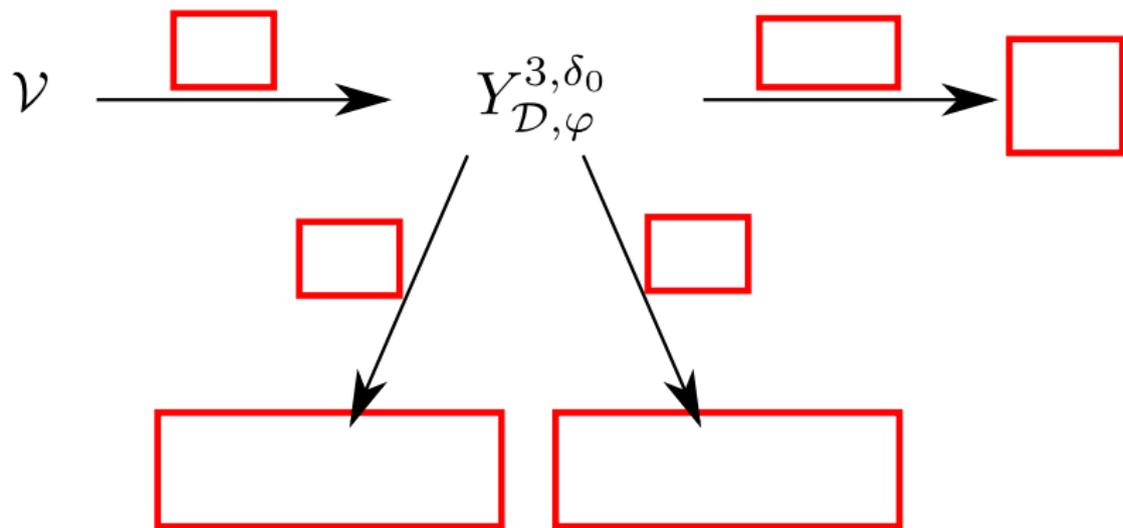


$$\mathfrak{Z} = [0, 1) \times Z_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$$

i.e. elements of the form

$$(r, \tilde{u}) \text{ with } \tilde{u} = (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)$$

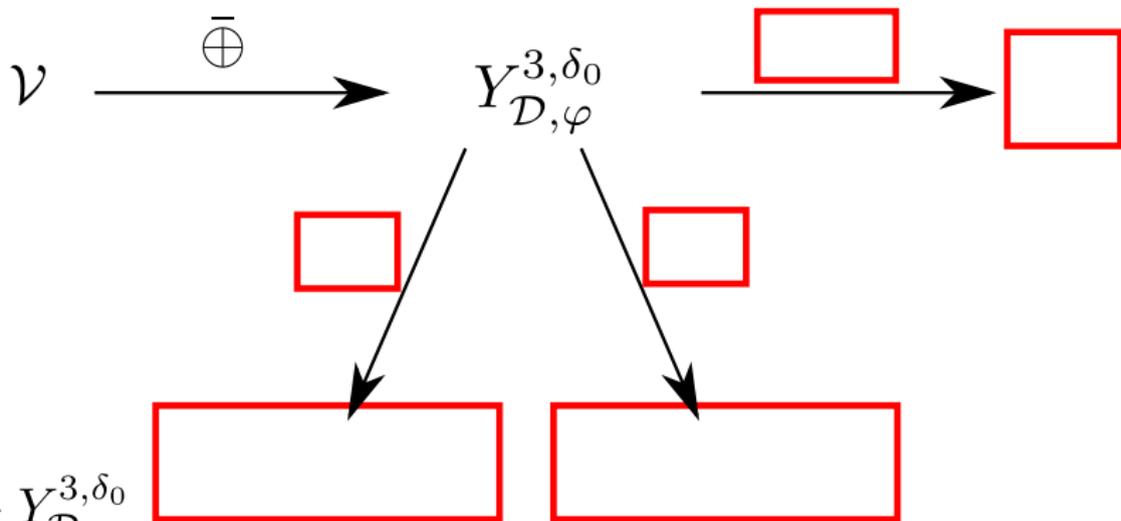
The periodic orbit case



$\mathcal{V} = \{(r, \tilde{u}) \in \mathfrak{Z} : \text{either } r = 0, \text{ or else } r > 0 \text{ and } (*) \text{ holds}\}$

$$(*) \quad \begin{aligned} & \varphi(r) + c^y - c^x > 0 \\ & \varphi^{-1}\left(\frac{1}{T} \cdot (\varphi(r) + c^x - c^y)\right) \in \left(0, \frac{1}{4}\right) \end{aligned}$$

The periodic orbit case



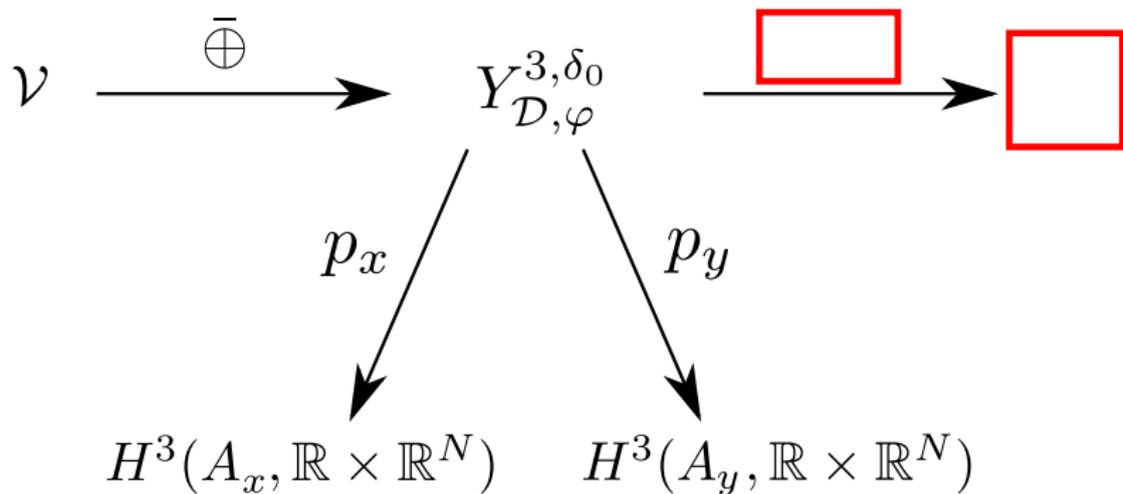
$$\bar{\oplus} : \mathcal{V} \rightarrow Y_{\mathcal{D}, \varphi}^{3, \delta_0}$$

$$\bar{\oplus}(0, (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)) = (0, (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y))$$

$$\bar{\oplus}(r, (\tilde{u}^x, [\hat{x}, \hat{y}], \tilde{u}^y)) = (r, \oplus_a(\tilde{u}^x, (\varphi(r) * \tilde{u}^y)))$$

where $a = |a| \cdot [\hat{x}, \hat{y}] \quad T \cdot \varphi(|a|) = \varphi(r) + c^y - c^x$

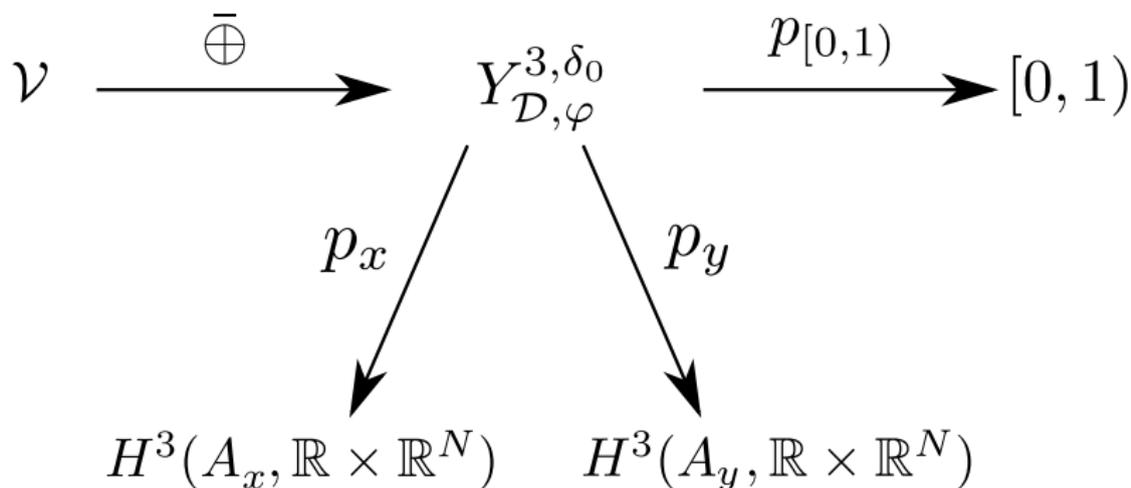
The periodic orbit case



where

$$p_x(r, \tilde{w}) = \tilde{w}|_{A_x} \quad p_y(r, \tilde{w}) = ((-\varphi(r)) * \tilde{w})|_{A_y}$$

The periodic orbit case



Theorem:

$(\bar{\oplus}, \{p_x, p_y\}, p_{[0,1)})$ is a
~~subversive imprinting with restrictions.~~
LEGO block.

The periodic orbit case

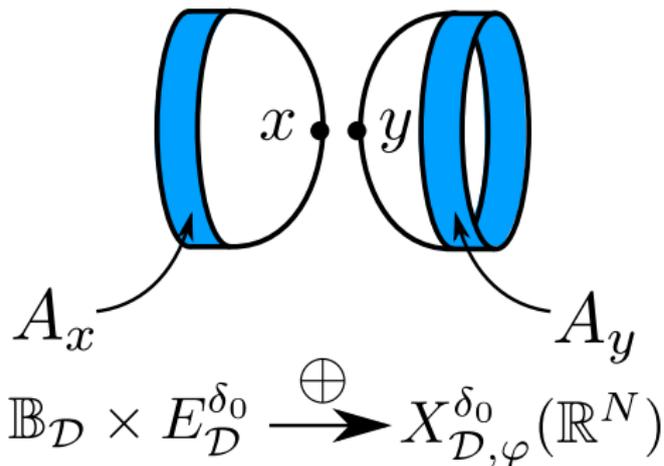
$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\bar{\oplus}} & Y_{\mathcal{D}, \varphi}^{3, \delta_0} & \xrightarrow{p_{[0,1]}} & [0, 1) \\ & & \swarrow p_x & & \searrow p_y \\ & & H^3(A_x, \mathbb{R} \times \mathbb{R}^N) & & H^3(A_y, \mathbb{R} \times \mathbb{R}^N) \end{array}$$

Theorem:

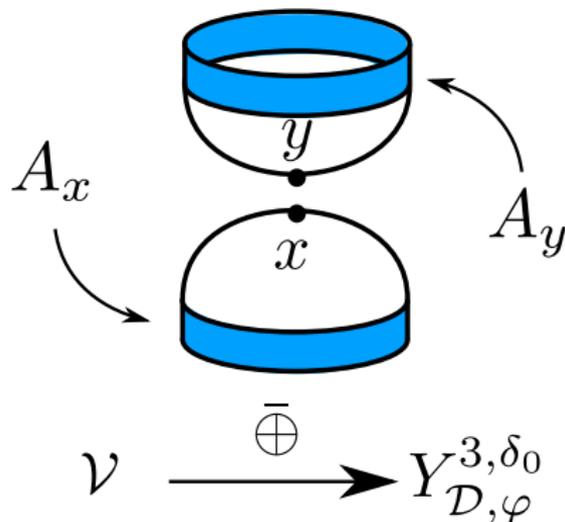
There is a functorial construction which extends to targets $\mathbb{R} \times Q$ from $\mathbb{R} \times \mathbb{R}^N$

Three important cases

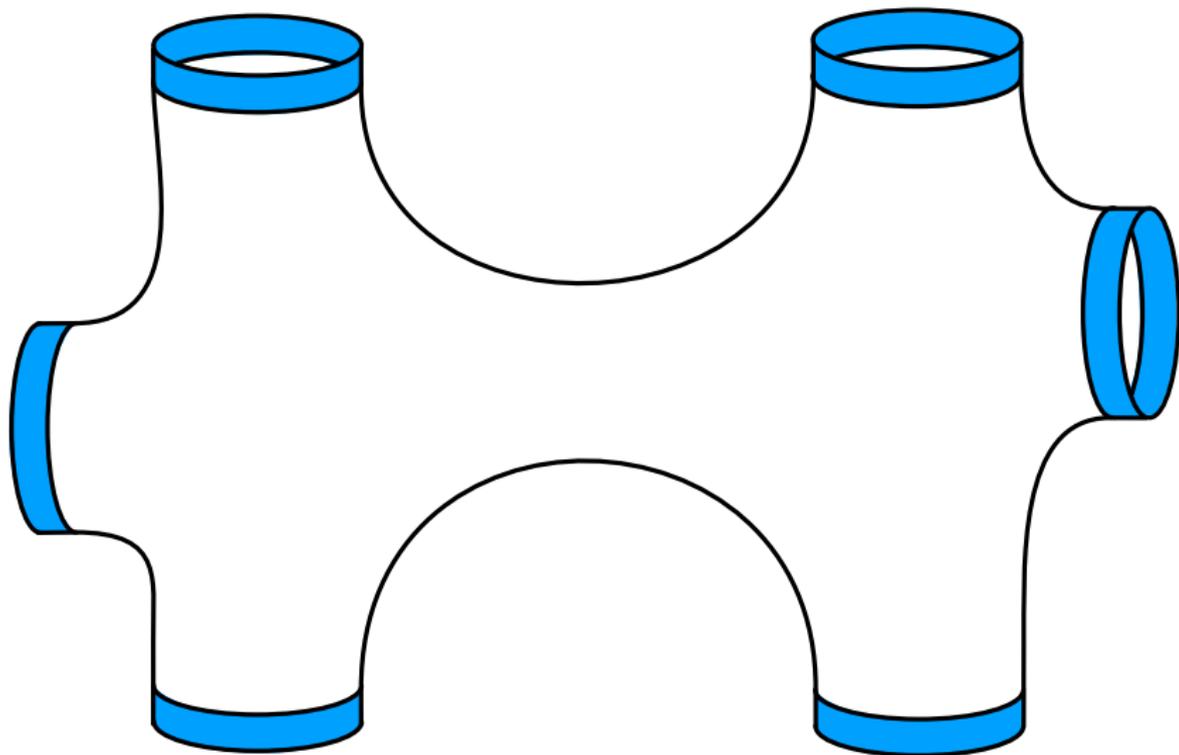
$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$



$$\mathcal{D} = (D_x \sqcup D_y, (x, y))$$

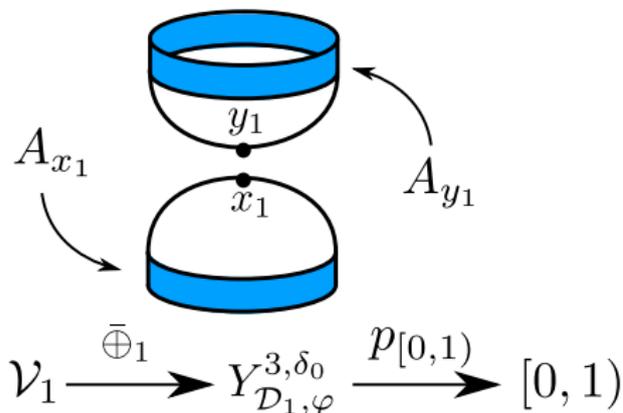


Three important cases

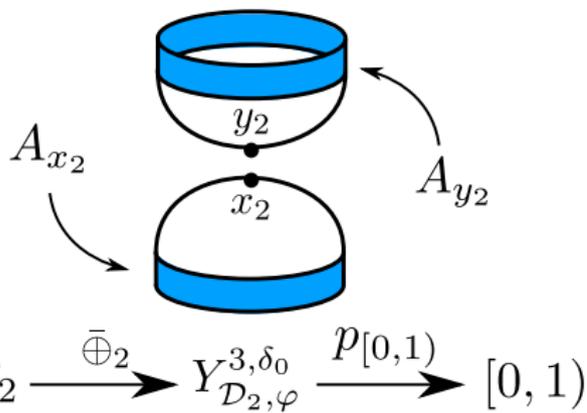


Three important cases

$$\mathcal{D}_1 = (D_{x_1} \sqcup D_{y_1}, (x_1, y_1))$$



$$\mathcal{D}_2 = (D_{x_2} \sqcup D_{y_2}, (x_2, y_2))$$

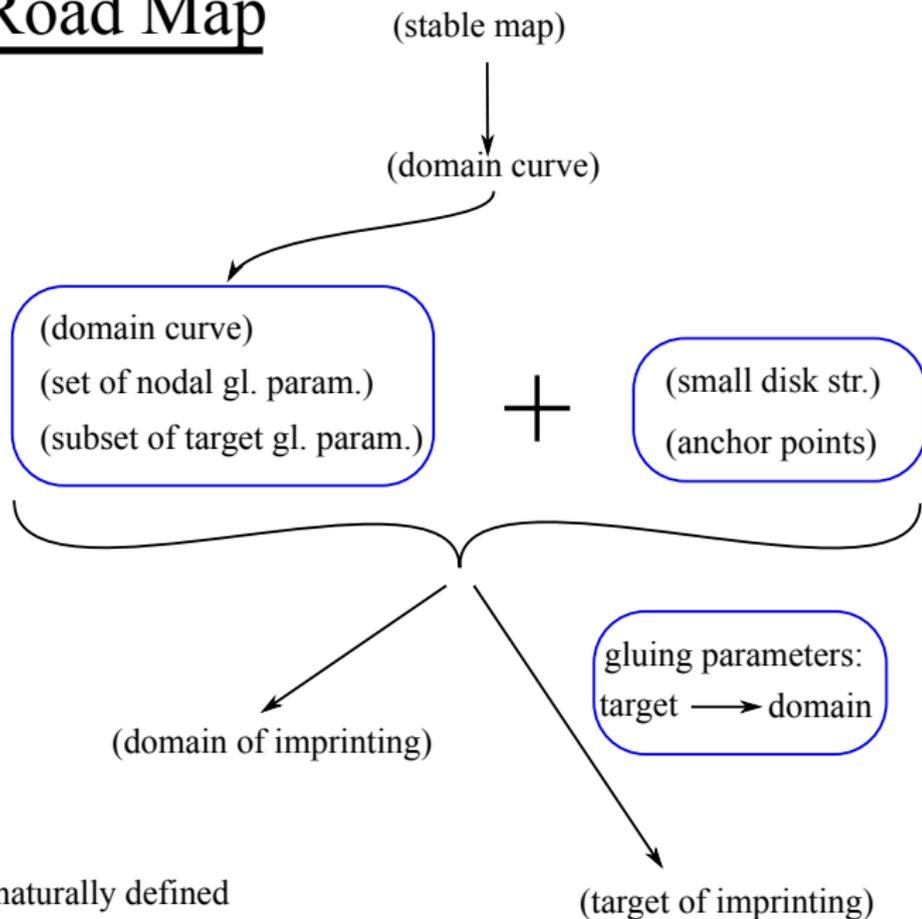


$$\mathcal{V}_1 \times \mathcal{V}_2 \xrightarrow{\bar{\oplus}_1 \times \bar{\oplus}_2} Y_{\mathcal{D}_1, \varphi}^{3, \delta_0} \times Y_{\mathcal{D}_2, \varphi}^{3, \delta_0} \xrightarrow{P_{[0, 1)} \times P_{[0, 1)}} [0, 1) \times [0, 1)$$

Need the pull back of this diagram by Δ

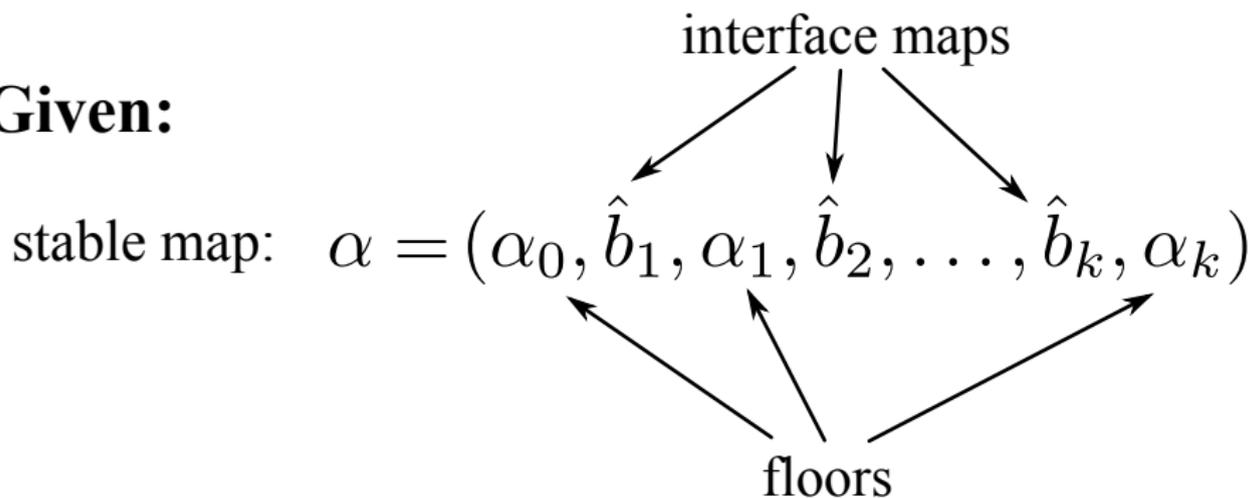
$$\begin{array}{c} \uparrow \Delta \\ [0, 1) \end{array}$$

Building -- Road Map



Data Preparation:

Given:



floor: $\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$

automorphism group preserving floor structure:

G

Data Preparation:

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

(S_i, j_i) Riemann surface

M_i marked points

D_i nodal pairs

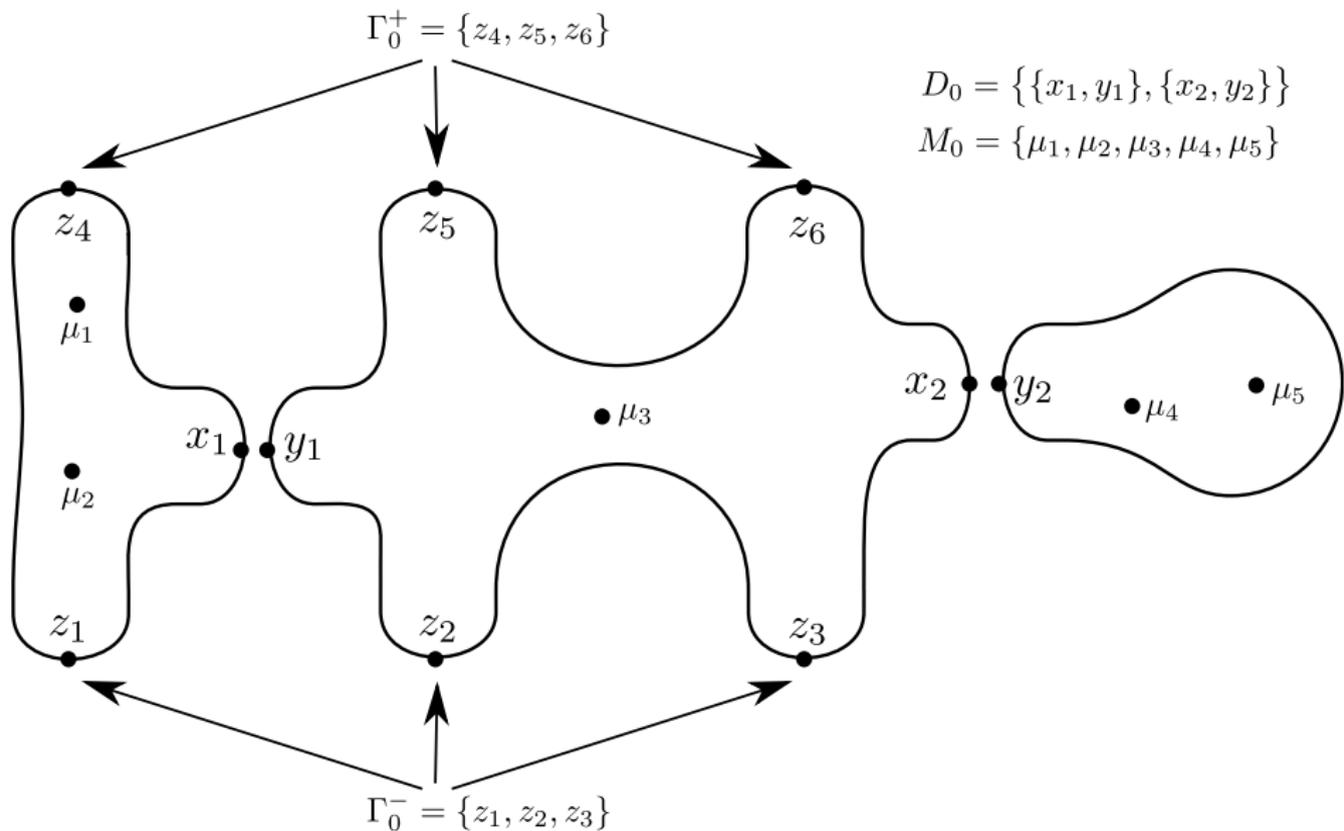
Γ_i^- negative punctures

Γ_i^+ positive punctures

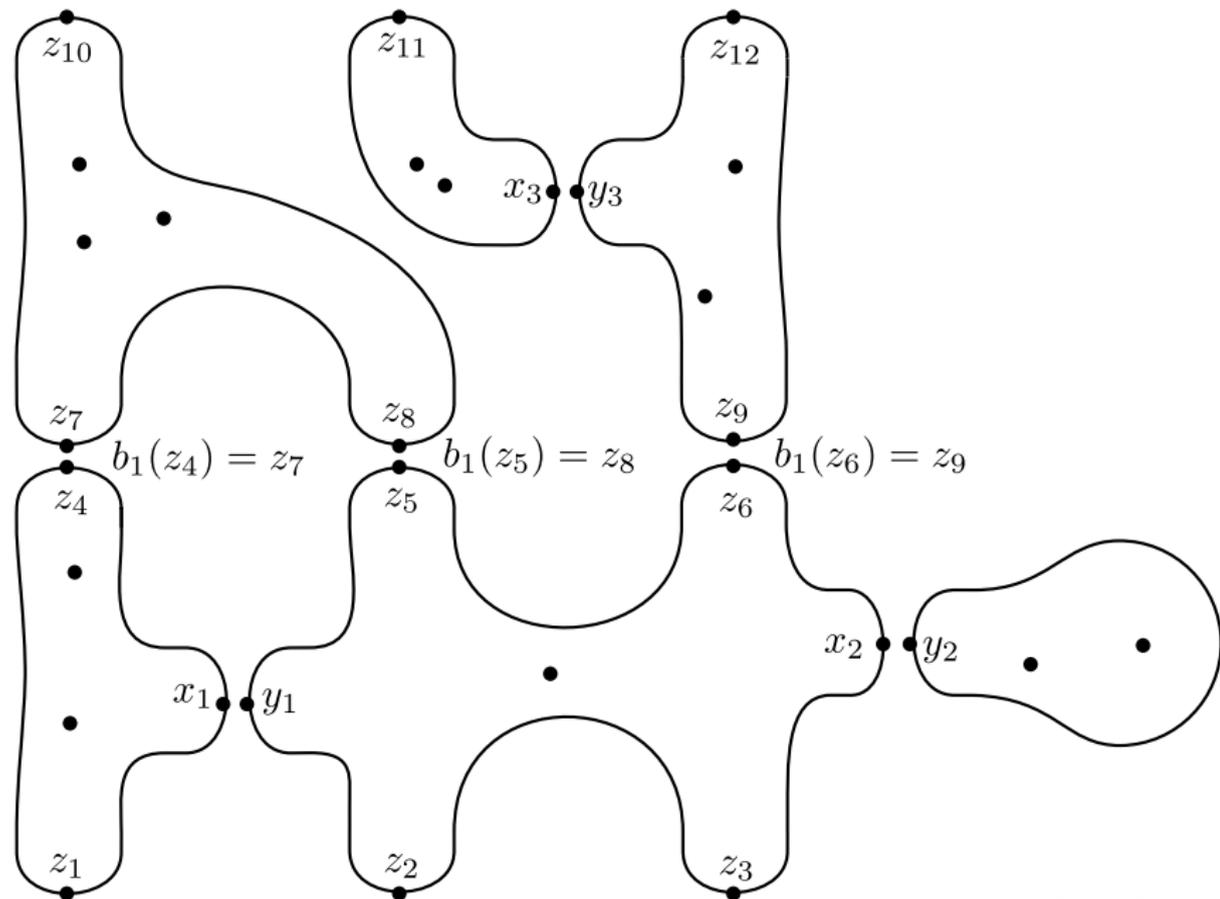
$[\tilde{u}_i]$ eq. class of maps

$$(a_i, u_i) = \tilde{u}_i \sim c * \tilde{u}_i = (a_i + c, u_i)$$

Data Preparation:



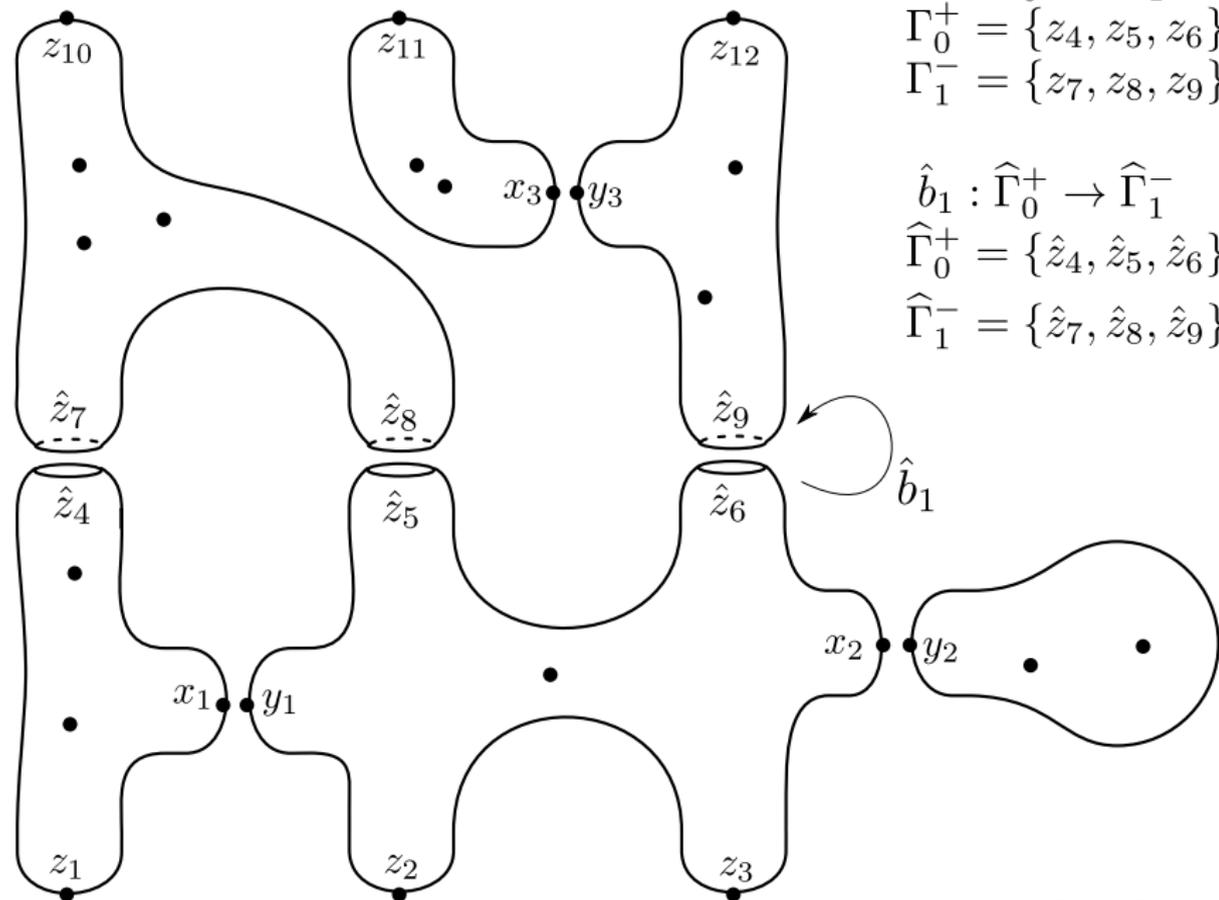
Data Preparation:



Data Preparation:

$$b_1 : \Gamma_0^+ \rightarrow \Gamma_1^-$$
$$\Gamma_0^+ = \{z_4, z_5, z_6\}$$
$$\Gamma_1^- = \{z_7, z_8, z_9\}$$

$$\hat{b}_1 : \hat{\Gamma}_0^+ \rightarrow \hat{\Gamma}_1^-$$
$$\hat{\Gamma}_0^+ = \{\hat{z}_4, \hat{z}_5, \hat{z}_6\}$$
$$\hat{\Gamma}_1^- = \{\hat{z}_7, \hat{z}_8, \hat{z}_9\}$$



Building -- Road Map

✓ (stable map)

(domain curve with floor str.)

(domain curve)
(set of nodal gl. param.)
(subset of target gl. param.)

+

(small disk str.)
(anchor points)

(domain of imprinting)

gluing parameters:
target \longrightarrow domain

(target of imprinting)

imprinting map naturally defined

Data Preparation:

$$\alpha = (\alpha_0, \hat{b}_1, \alpha_1, \hat{b}_2, \dots, \hat{b}_k, \alpha_k)$$



$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \dots, b_k, \sigma_k)$$

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

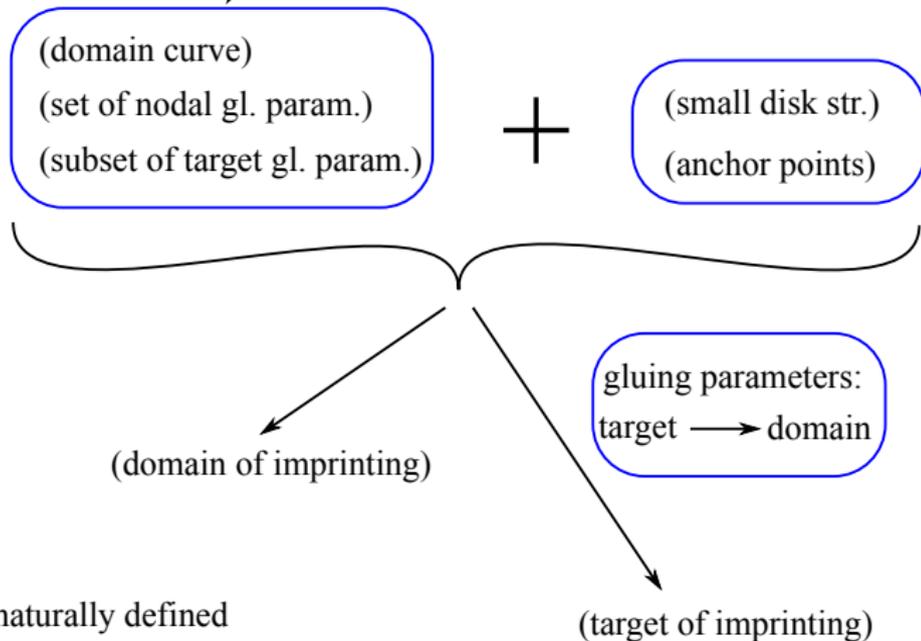


$$\sigma_i = (\Gamma_i^-, S_i, j_i, D_i, \Gamma_i^+)$$

Building -- Road Map

✓ (stable map)

✓ (domain curve with floor str.)



imprinting map naturally defined

Data Preparation:

$$\alpha = (\alpha_0, \hat{b}_1, \alpha_1, \hat{b}_2, \dots, \hat{b}_k, \alpha_k)$$



$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \dots, b_k, \sigma_k)$$

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$



$$\sigma_i = (\Gamma_i^-, S_i, j_i, D_i, \Gamma_i^+)$$

add small disk structures:

$$\mathbf{D}_i^+ \text{ about } \Gamma_i^+ \quad i \in \{0, 1, \dots, k-1\}$$

$$\mathbf{D}_i \text{ about } |D_i| \quad i \in \{0, 1, \dots, k\}$$

$$\mathbf{D}_i^- \text{ about } \Gamma_i^- \quad i \in \{1, 2, \dots, k\}$$

add anchor points:

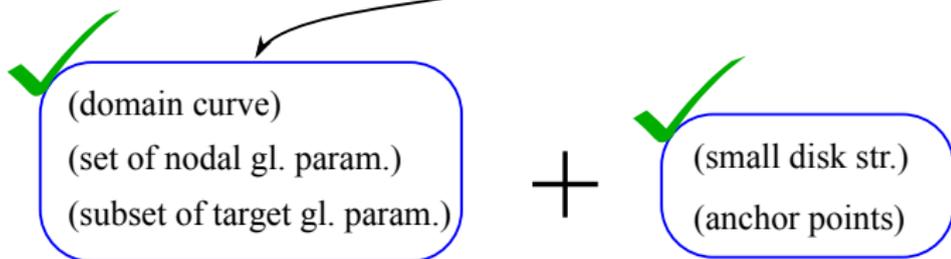
$$\mathcal{J}_i = \mathcal{J} \cap S_i \neq \emptyset$$

all data G invariant

Building -- Road Map

✓ (stable map)

✓ (domain curve with floor str.)



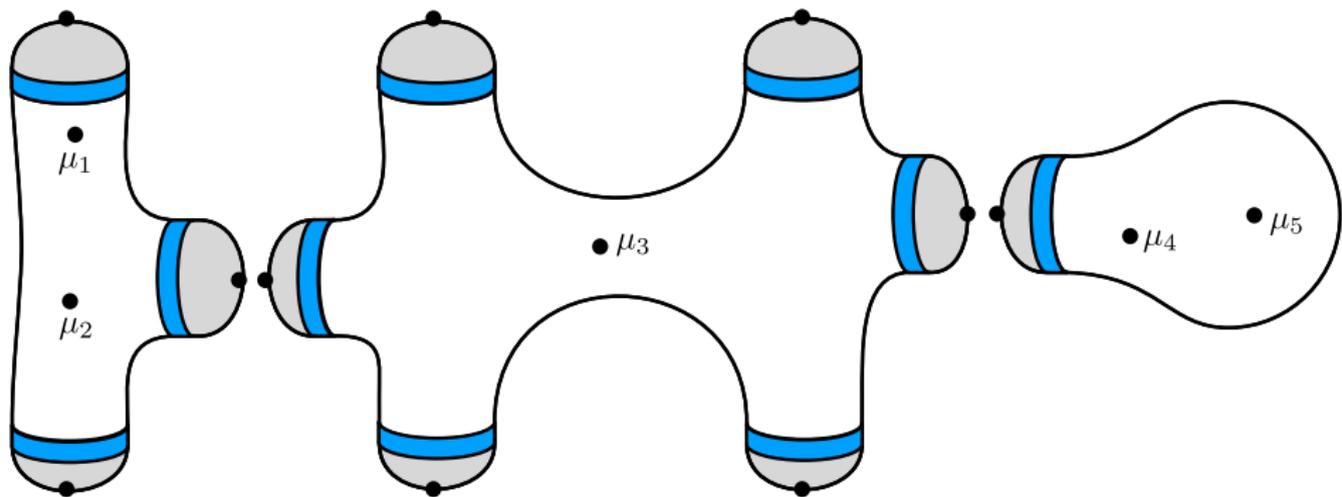
(domain of imprinting)

gluing parameters:
target \longrightarrow domain

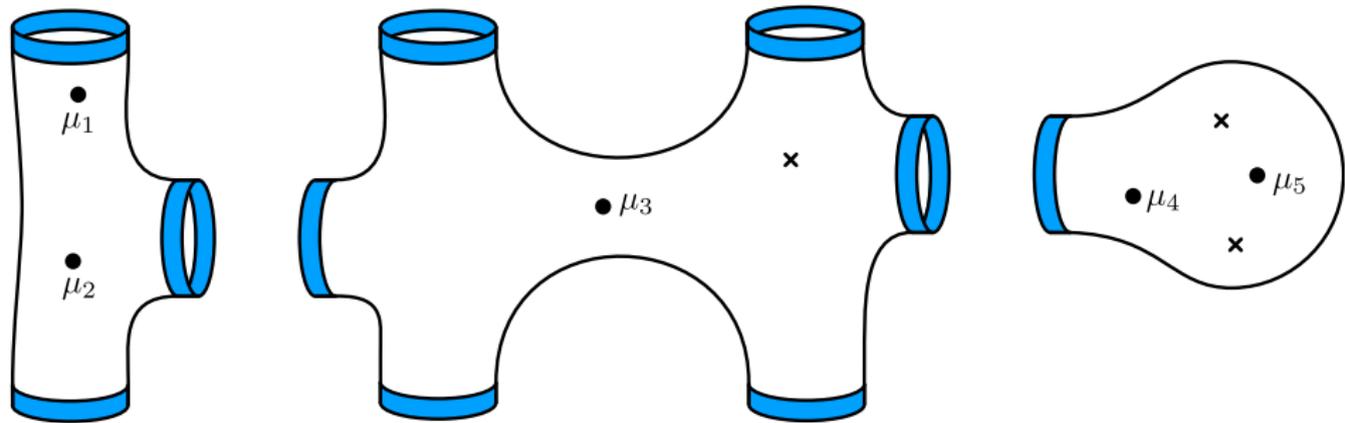
imprinting map naturally defined

(target of imprinting)

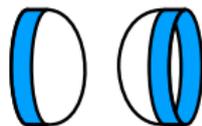
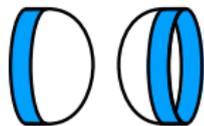
Fragmentation -- Focus on a floor



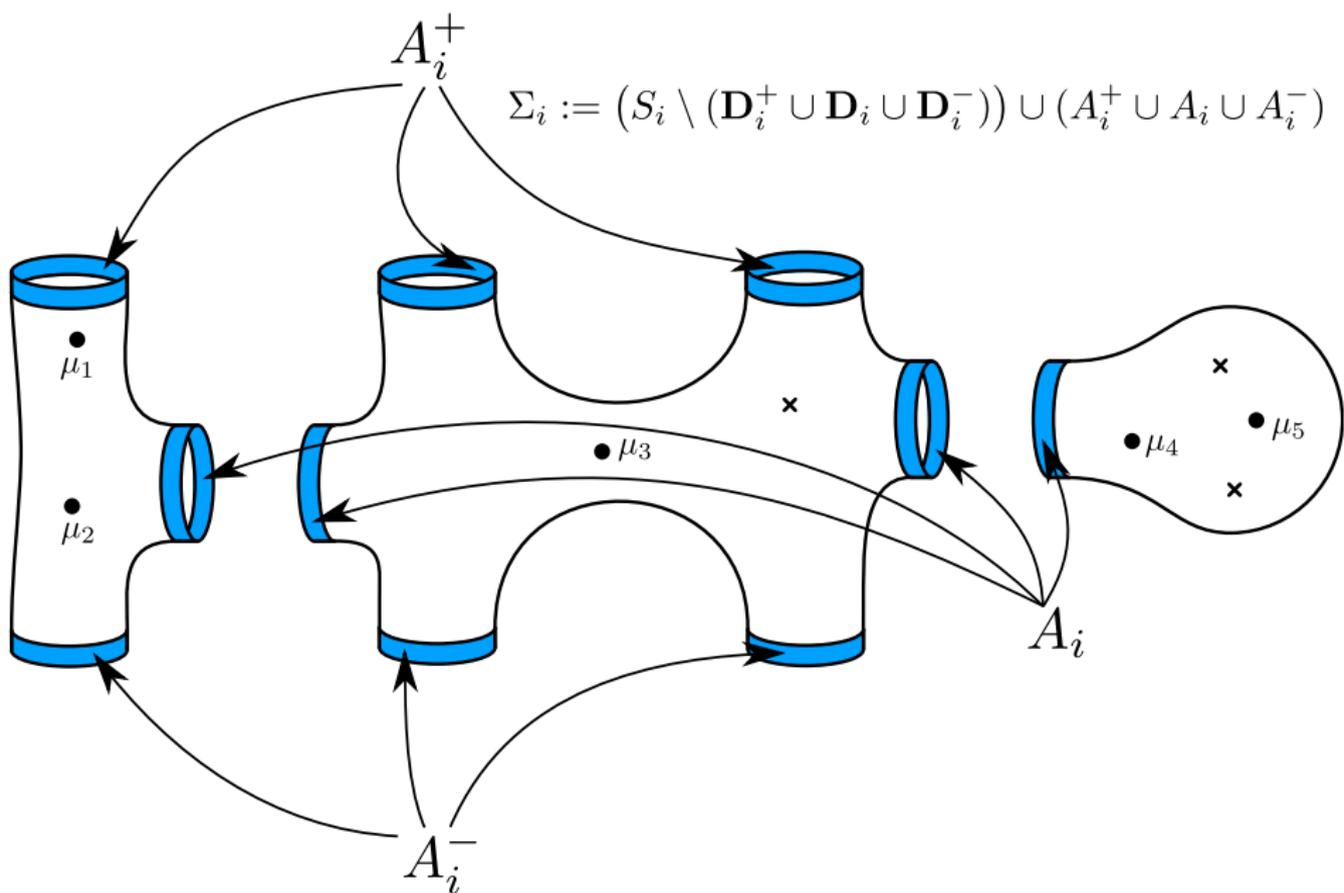
Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor

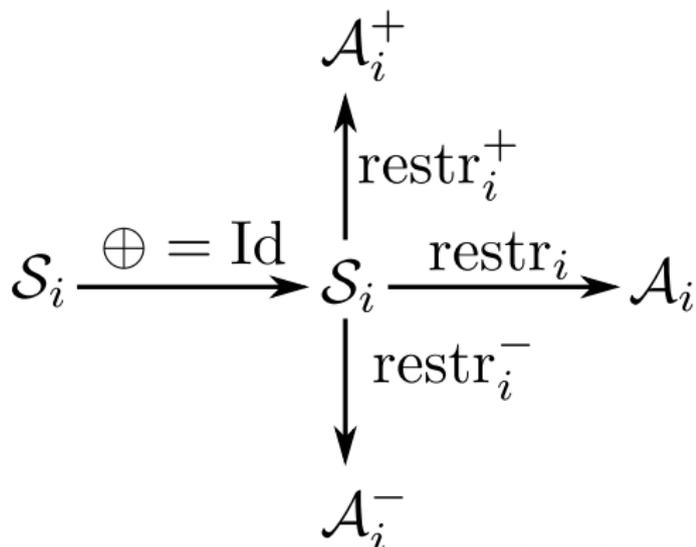
$$\mathcal{S}_i = \{ \tilde{u} \in H^3(\Sigma_i) : \text{av}_{\mathcal{L}_i}(\tilde{u}) = 0 \}$$

$$\mathcal{A}_i^- = H^3(A_i^-, \mathbb{R} \times \mathbb{R}^N)$$

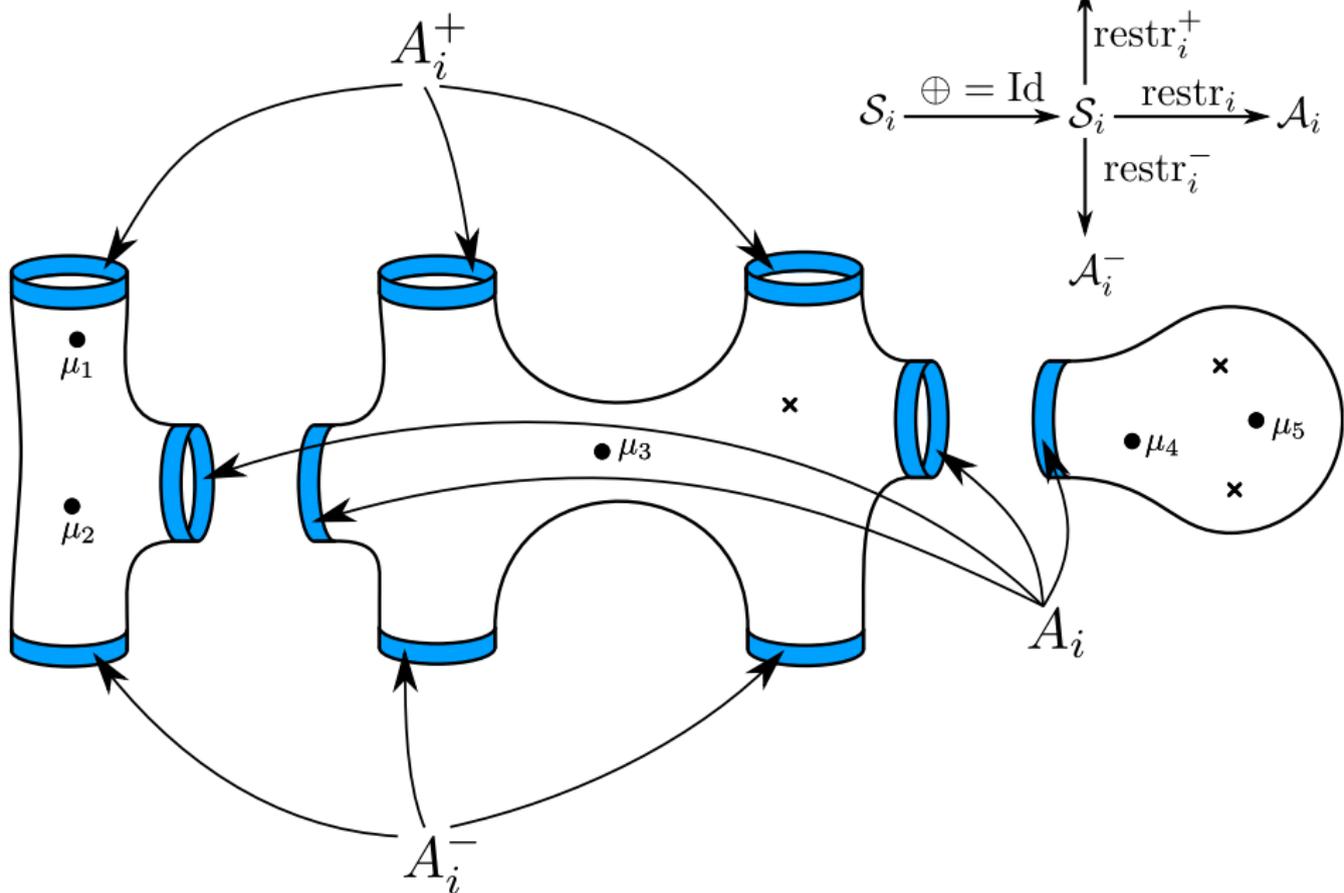
$$\mathcal{A}_i = H^3(A_i, \mathbb{R} \times \mathbb{R}^N)$$

$$\mathcal{A}_i^+ = H^3(A_i^+, \mathbb{R} \times \mathbb{R}^N)$$

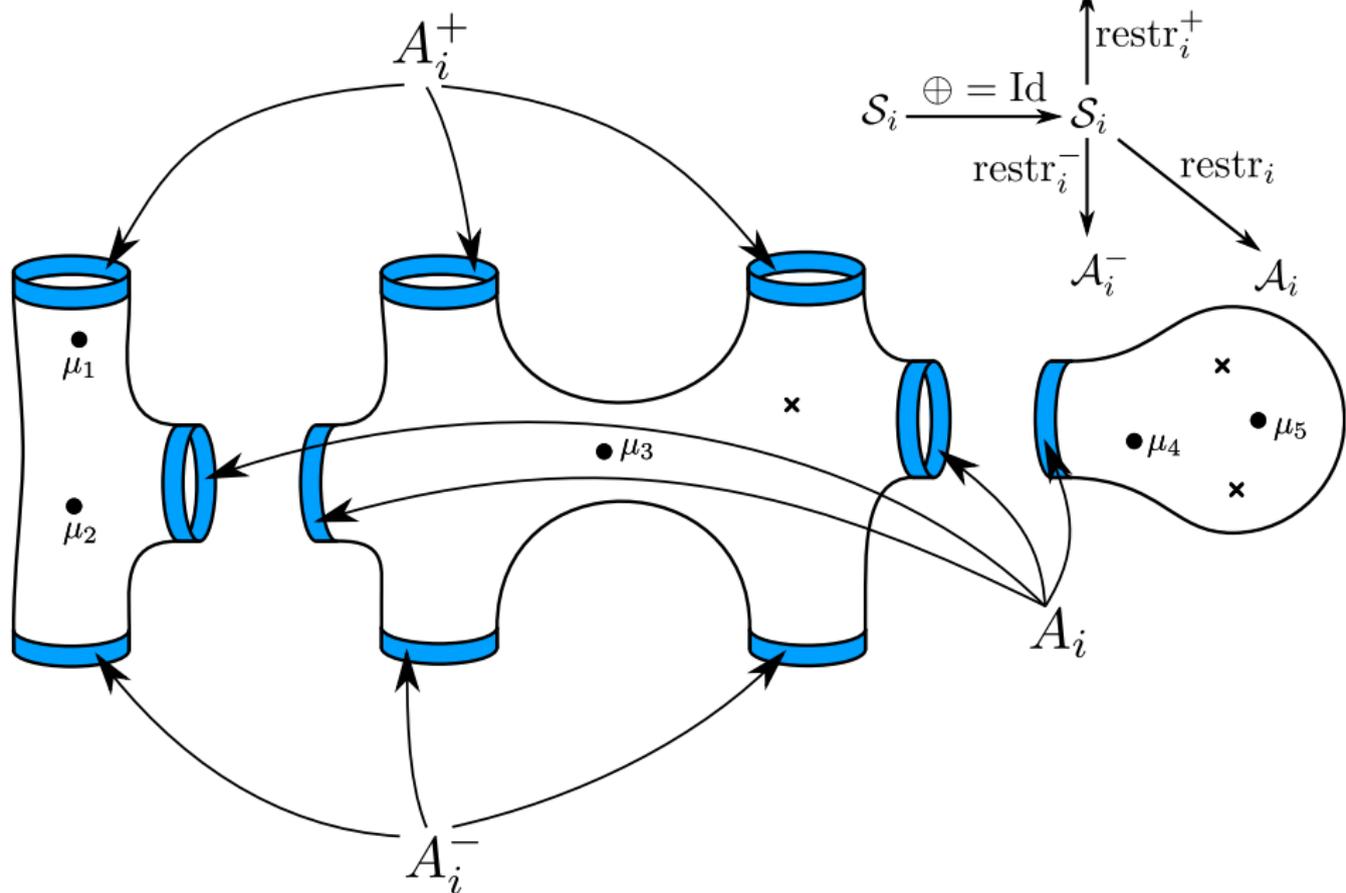
LEGO block



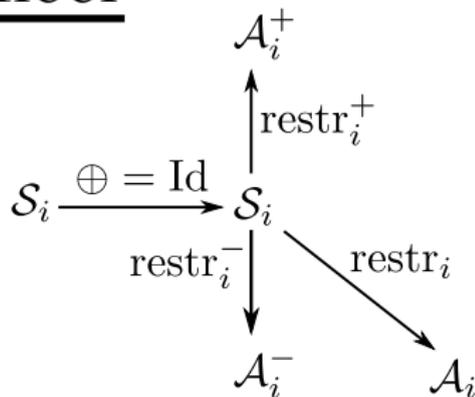
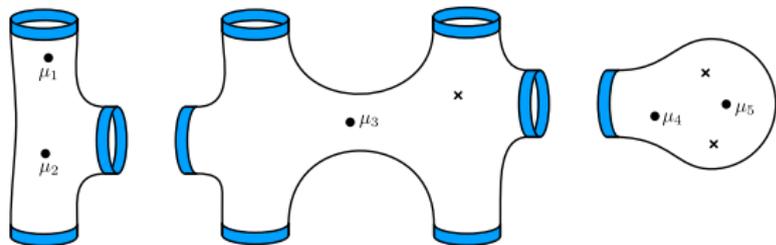
Fragmentation -- Focus on a floor



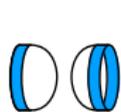
Fragmentation -- Focus on a floor



Fragmentation -- Focus on a floor



$$(\mathbb{B}_{\mathcal{D}_1} \times E_{\mathcal{D}_1}^{\delta_0}) \times (\mathbb{B}_{\mathcal{D}_2} \times E_{\mathcal{D}_2}^{\delta_0}) \xrightarrow{\oplus_1 \times \oplus_2} X_{\mathcal{D}_1, \varphi}^{\delta_0} \times X_{\mathcal{D}_2, \varphi}^{\delta_0} \xrightarrow{p_{\mathbb{B}_{\mathcal{D}_1}} \times p_{\mathbb{B}_{\mathcal{D}_2}}} \mathbb{B}_{\mathcal{D}_1} \times \mathbb{B}_{\mathcal{D}_2}$$



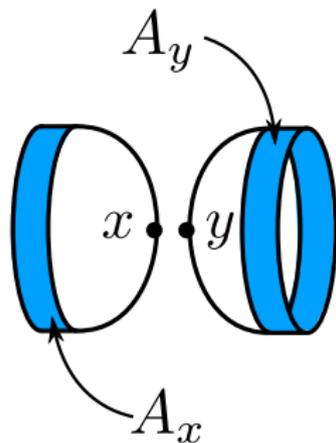
$$\begin{array}{c}
 \text{restr}_i \\
 \searrow \\
 \mathcal{A}_i
 \end{array}$$

Fragmentation -- Construct a building

Recall:

$$\mathcal{D} = (D_x \sqcup D_y, \{x, y\})$$

$$\begin{array}{ccc} \mathbb{B}_{\mathcal{D}} \times E_{\mathcal{D}}^{\delta_0} \xrightarrow{\oplus} & X_{\mathcal{D}, \varphi}^{\delta_0}(\mathbb{R}^N) & \xrightarrow{p_{\mathbb{B}_{\mathcal{D}}}} \mathbb{B}_{\mathcal{D}} \\ & \swarrow p_x & \searrow p_y \\ & H^3(A_x, \mathbb{R}^N) & H^3(A_y, \mathbb{R}^N) \end{array}$$



$$\underline{E_{\mathcal{D}}^{\delta_0}} = \mathbb{R}^N \oplus H^{3, \delta_0}(D_x \sqcup D_y, \mathbb{R}^N)$$

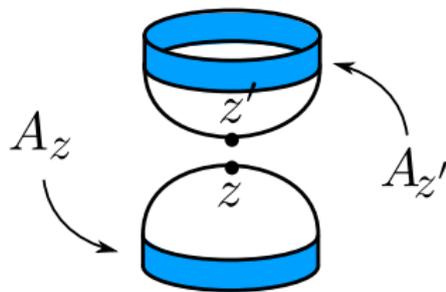
Fragmentation -- Construct a building

Recall:

$$\mathcal{V} \xrightarrow{\bar{\oplus}} Y_{\mathcal{D}, \varphi}^{3, \delta_0} \xrightarrow{p_{[0,1]}} [0, 1)$$

$$\begin{array}{ccc} & & \\ & \swarrow p_z & \searrow p_{z'} \\ H^3(A_z, \mathbb{R} \times \mathbb{R}^N) & & H^3(A_{z'}, \mathbb{R} \times \mathbb{R}^N) \end{array}$$

$$\mathcal{D} = (D_z \sqcup D_{z'}, (z, z'))$$



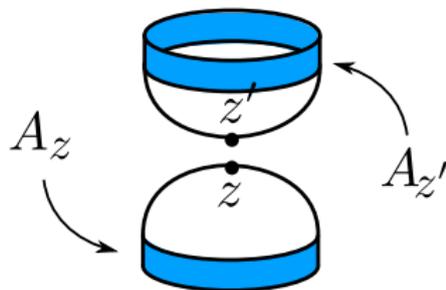
Fragmentation -- Construct a building

Recall:

$$\mathcal{V} \xrightarrow{\bar{\oplus}} Y_{\mathcal{D}, \varphi}^{3, \delta_0} \xrightarrow{p_{[0,1]}} [0, 1)$$

$$\begin{array}{ccc} & & \\ & \swarrow p_z & \searrow p_{z'} \\ & H^3(A_z, \mathbb{R} \times \mathbb{R}^N) & H^3(A_{z'}, \mathbb{R} \times \mathbb{R}^N) \end{array}$$

$$\mathcal{D} = (D_z \sqcup D_{z'}, (z, z'))$$



$$\mathcal{V} \subset [0, 1) \times \underline{Z}_{\mathcal{D}}(\mathbb{R} \times \mathbb{R}^N, \bar{\gamma})$$

elements of the form $\tilde{u} = (\tilde{u}^z, [\hat{x}, \hat{y}], \tilde{u}^{z'})$

$$\tilde{u}^z \circ \sigma_{\hat{x}}^+(s, t) = (Ts + c^z, \gamma(kt)) + \tilde{r}^z(s, t)$$

$$\tilde{u}^{z'} \circ \sigma_{\hat{y}}^-(s', t') = (Ts' + c^{z'}, \gamma(kt')) + \tilde{r}^{z'}(s', t')$$

here $\tilde{r}^z, \tilde{r}^{z'} \in H^{3, \delta_0}$

Recall:

$$\Sigma_i := (S_i \setminus (\mathbf{D}_i^+ \cup \mathbf{D}_i \cup \mathbf{D}_i^-)) \cup (A_i^+ \cup A_i \cup A_i^-)$$

$$\mathcal{S}_i := \{ \tilde{u} \in H^3(\Sigma_i) : \text{av}_{\mathcal{J}_i}(\tilde{u}) = 0 \}$$

$$\text{av}_{\mathcal{J}_i}(\tilde{u}) := \frac{1}{\#\mathcal{J}_i} \cdot \sum_{z \in \mathcal{J}_i} a_i(z)$$

$$\Gamma := \bigcup_{i=0}^k (\Gamma_i^+ \cup \Gamma_i^-)$$

$$\bar{\mathbf{F}} : \Gamma \rightarrow \{ \tilde{\gamma} : \text{weighted periodic orbit in } \mathbb{R}^N \}$$

which satisfies $\bar{\mathbf{F}}(z) = \bar{\mathbf{F}}(b_i(z))$ for each $z \in \Gamma_i^+$ and $i \in \{0, \dots, k-1\}$

Define ssc-Hilbert manifold $Z_{\sigma, \mathcal{J}}^3(\mathbb{R} \times \mathbb{R}^N, \bar{\mathbf{F}})$

$$Z_{\sigma, \mathcal{J}}^3(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}}) =$$

(truncated) floor 0	$Z_{\mathcal{D}_0}^- \times_{\mathcal{A}_0^-}$ $\mathcal{S}_0 \times_{\mathcal{A}_0} E_{\mathcal{D}_0}$	Negative ends of bottom level
(truncated) floor 1	$\times_{\mathcal{A}_0^+} Z_{\mathcal{D}_1} \times_{\mathcal{A}_1^-}$ $\mathcal{S}_1 \times_{\mathcal{A}_1} E_{\mathcal{D}_1}$	Interface level 1
(truncated) floor 2	$\times_{\mathcal{A}_1^+} Z_{\mathcal{D}_2} \times_{\mathcal{A}_2^-}$ $\mathcal{S}_2 \times_{\mathcal{A}_2} E_{\mathcal{D}_2}$	Interface level 2
	$\times_{\mathcal{A}_2^+} Z_{\mathcal{D}_3} \times_{\mathcal{A}_3^-}$	Interface level 3
	⋮	
(truncated) floor k	$\times_{\mathcal{A}_{k-1}^+} Z_{\mathcal{D}_k} \times_{\mathcal{A}_k^-}$ $\mathcal{S}_k \times_{\mathcal{A}_k} E_{\mathcal{D}_k}$	Interface level k
	$\times_{\mathcal{A}_k^+} Z_{\mathcal{D}_k}^+$	Positive ends of top level

The takeaway:

$Z_{\sigma, \mathcal{J}}^3(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}})$ is an ssc-manifold consisting of tuples of the form

$$\tilde{u} := (\tilde{u}_0, \hat{b}_1, \dots, \hat{b}_k, \tilde{u}_k)$$

where each \tilde{u}_i is of class $(3, \delta_0)$ and asymptotic to the weighted periodic orbits prescribed by $\overline{\mathbf{F}}$ so that the data across interfaces is \hat{b}_i matching, and the anchor averages vanish.

Domain of Imprinting (almost):

ssc-manifold
just constructed

$$\mathbb{B}_D \times [0, 1)^k \times Z_{\sigma, \mathcal{J}}^3(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}})$$

tuple of target gluing parameters
one for each interface level

product of domain gluing parameters $\mathbb{B}_{D_{\{x,y\}}}$
for each $\{x, y\} \in D$

Domain of Imprinting (actual):

$$\underline{\mathbb{B}_{\mathcal{D}} \times \mathcal{O}}$$

where $\mathcal{O} \subset [0, 1]^k \times Z_{\sigma, \mathcal{J}}^3(\mathbb{R} \times \mathbb{R}^N, \overline{\mathbf{F}})$ consists of tuples $(r_1, \dots, r_k, \tilde{u})$ with $r_i \in (0, 1)$ and $\tilde{u} := (\tilde{u}_0, \hat{b}_1, \dots, \hat{b}_k, \tilde{u}_k)$ such that either

1. $r_i = 0$

2.
$$\begin{cases} \varphi(r_i) - c^z(\tilde{u}) + c^{b_i(z)}(\tilde{u}) > 0 \\ \varphi^{-1}\left(\frac{1}{T_z} \cdot (\varphi(r_i) - c^z(\tilde{u}) + c^{b_i(z)}(\tilde{u}))\right) \in (0, \frac{1}{4}) \end{cases}$$

Building -- Road Map

✓ (stable map)

✓ (domain curve with floor str.)

✓
(domain curve)
(set of nodal gl. param.)
(subset of target gl. param.)

+

✓
(small disk str.)
(anchor points)

✓
(domain of imprinting)

gluing parameters:
target \longrightarrow domain

imprinting map naturally defined

(target of imprinting)

$\sigma = (S, j, M, D)$

Let

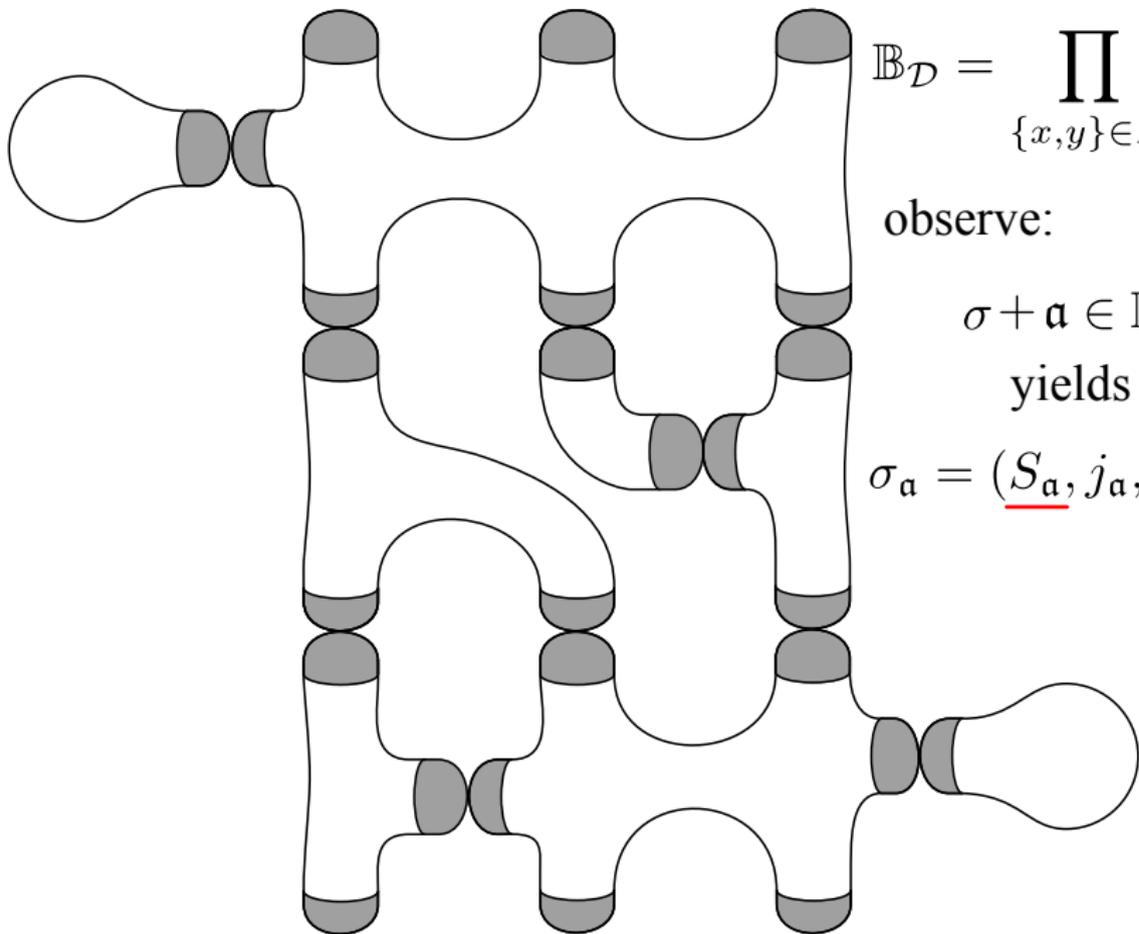
$$\mathbb{B}_D = \prod_{\{x,y\} \in D} \mathbb{B}_{\mathcal{D}_{\{x,y\}}}$$

observe:

$$\sigma + \mathbf{a} \in \mathbb{B}_D$$

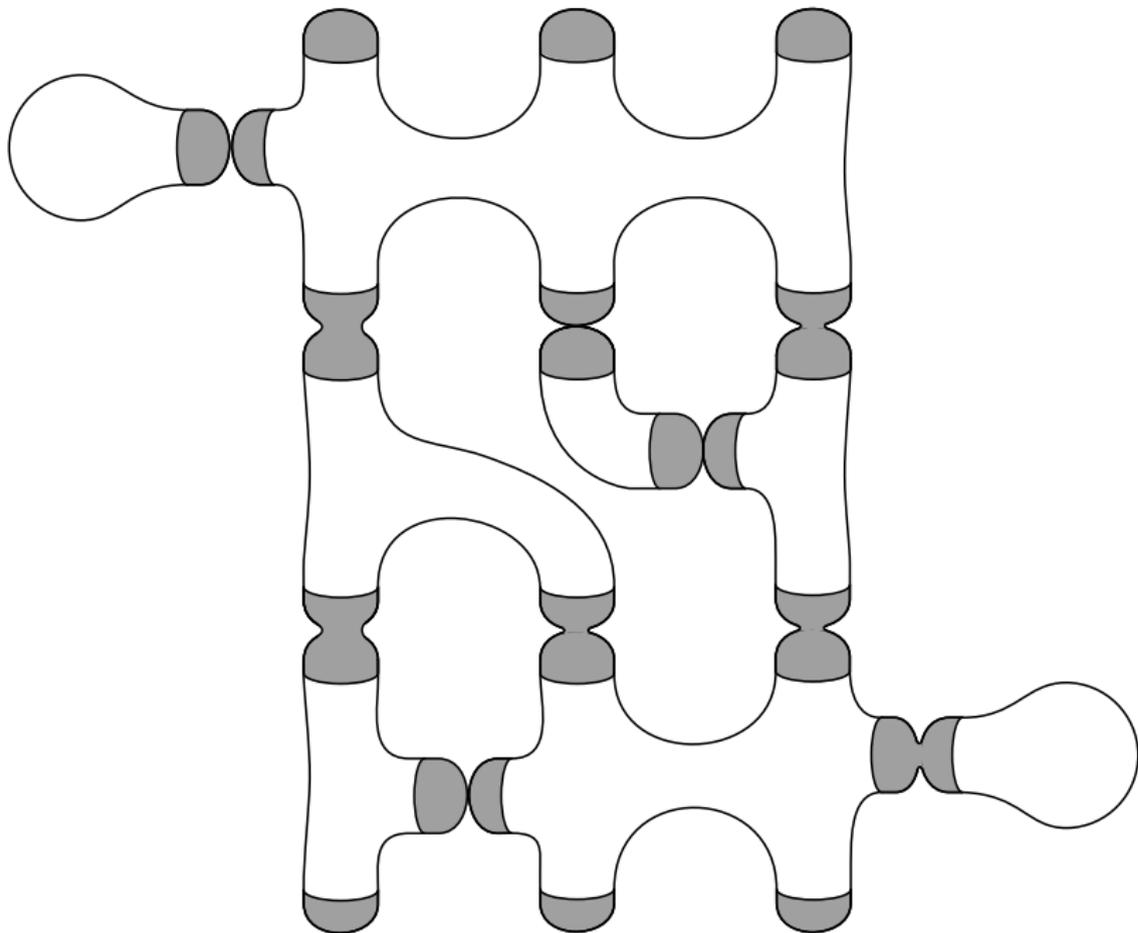
yields

$$\sigma_{\mathbf{a}} = (\underline{S_{\mathbf{a}}}, j_{\mathbf{a}}, M_{\mathbf{a}}, \underline{D_{\mathbf{a}}})$$



$$\sigma_{\mathbf{a}} = (S_{\mathbf{a}}, j_{\mathbf{a}}, M_{\mathbf{a}}, D_{\mathbf{a}})$$

$$\mathbf{a} \in \mathbb{B}_{\mathcal{D}}$$



$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \sigma_2)$$

Let

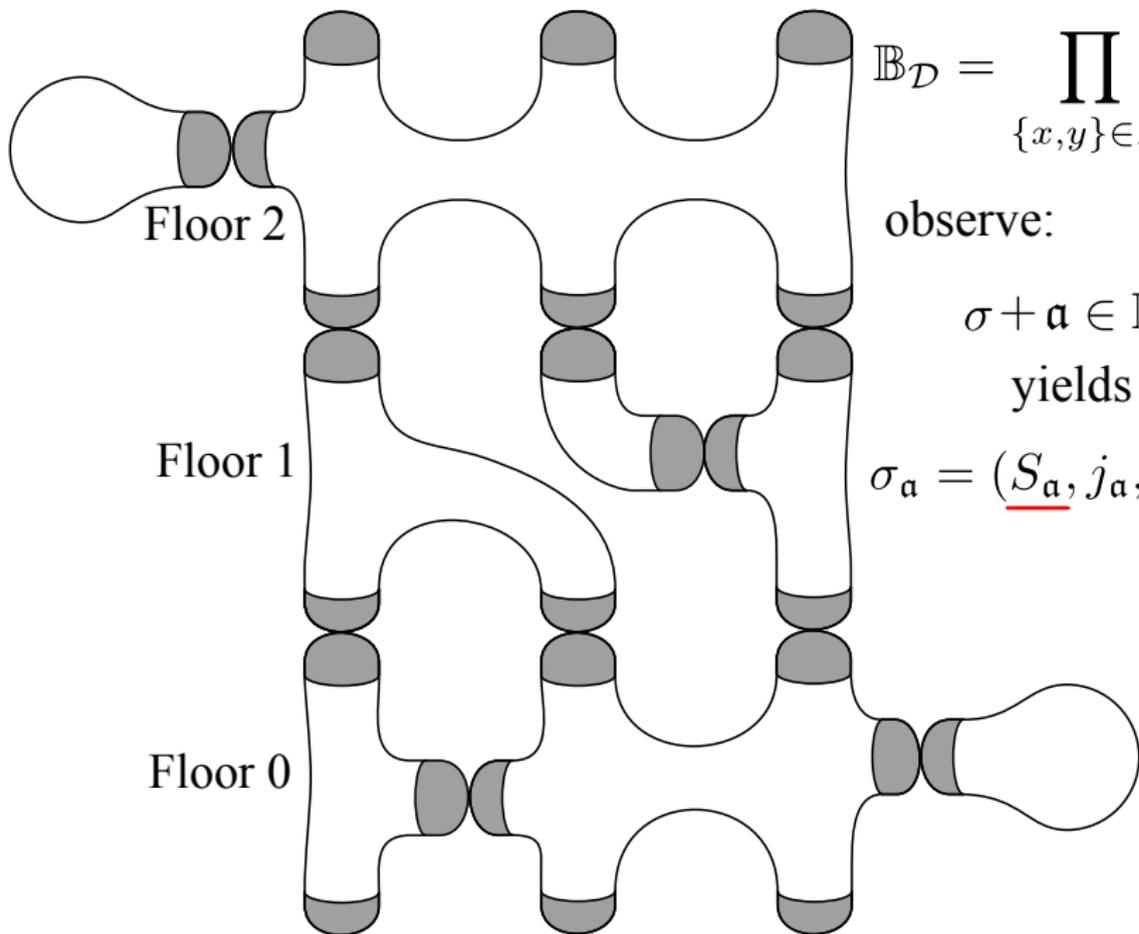
$$\mathbb{B}_D = \prod_{\{x,y\} \in D} \mathbb{B}_{\mathcal{D}\{x,y\}}$$

observe:

$$\sigma + \mathbf{a} \in \mathbb{B}_D$$

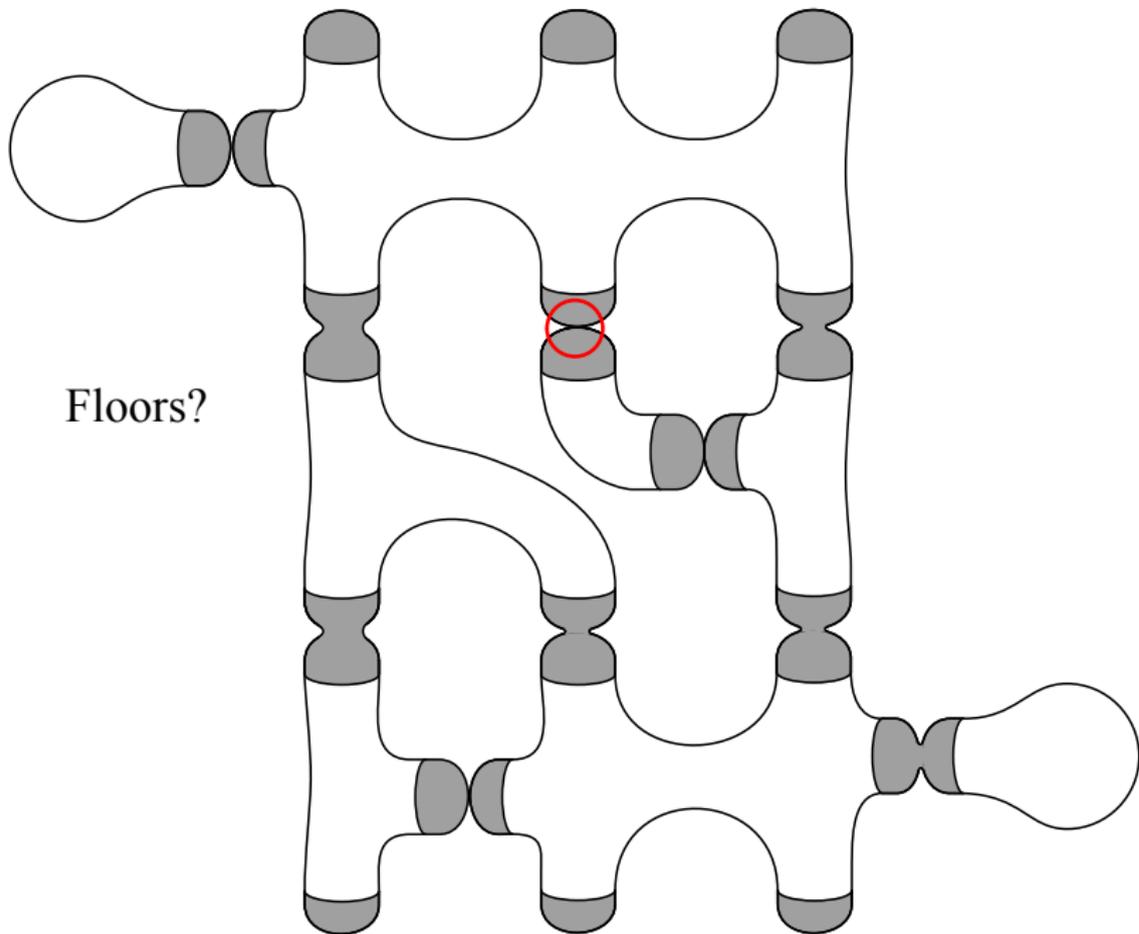
yields

$$\sigma_{\mathbf{a}} = (\underline{S_{\mathbf{a}}}, j_{\mathbf{a}}, M_{\mathbf{a}}, \underline{D_{\mathbf{a}}})$$



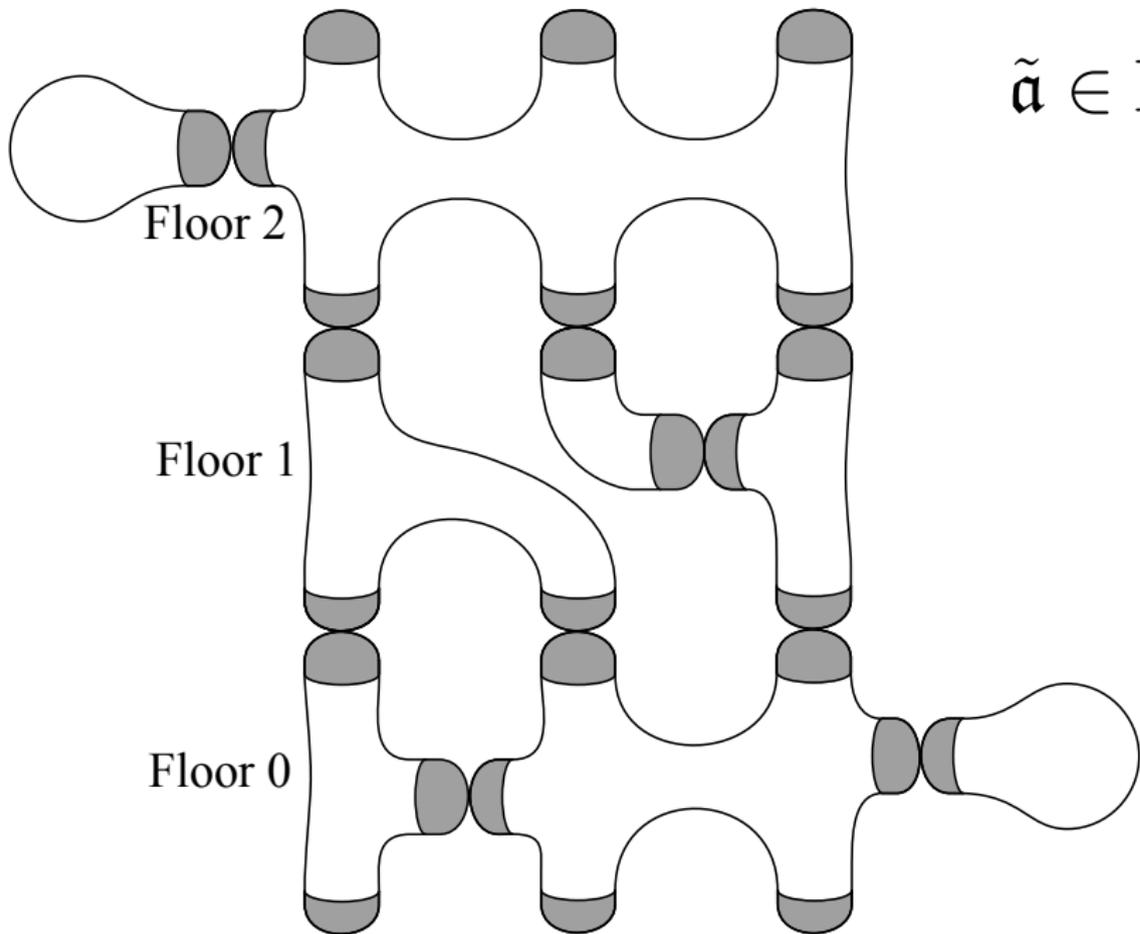
$\sigma_{\mathbf{a}} = \text{nonsense}$

$\mathbf{a} \in \mathbb{B}_{\mathcal{D}}$



$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \sigma_2)$$

$$\tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}}$$

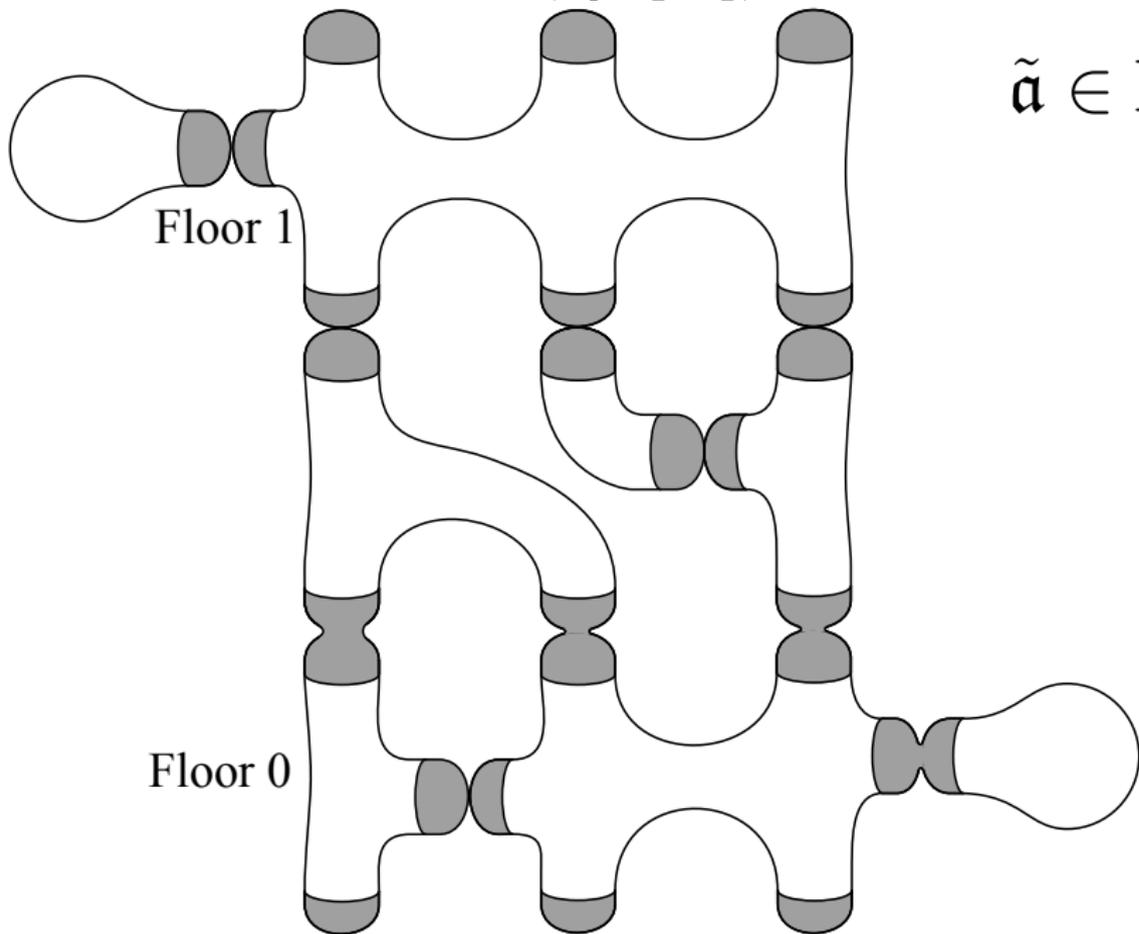


$$\sigma_{\tilde{\mathbf{a}}} = (\sigma_0^{\tilde{\mathbf{a}}}, b_1^{\tilde{\mathbf{a}}}, \sigma_1^{\tilde{\mathbf{a}}})$$

$$\tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}}$$

Floor 1

Floor 0

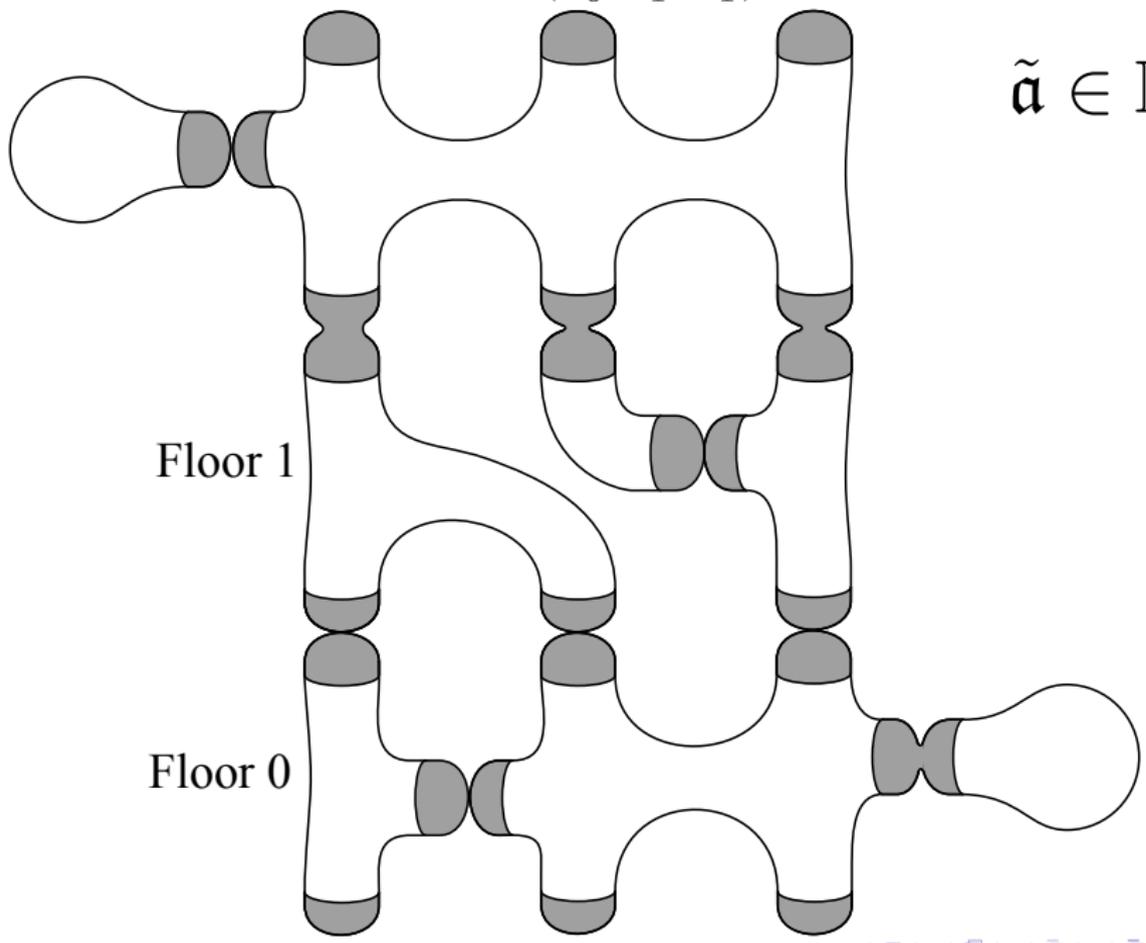


$$\sigma_{\tilde{\mathbf{a}}} = (\sigma_0^{\tilde{\mathbf{a}}}, b_1^{\tilde{\mathbf{a}}}, \sigma_1^{\tilde{\mathbf{a}}})$$

$$\tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}}$$

Floor 1

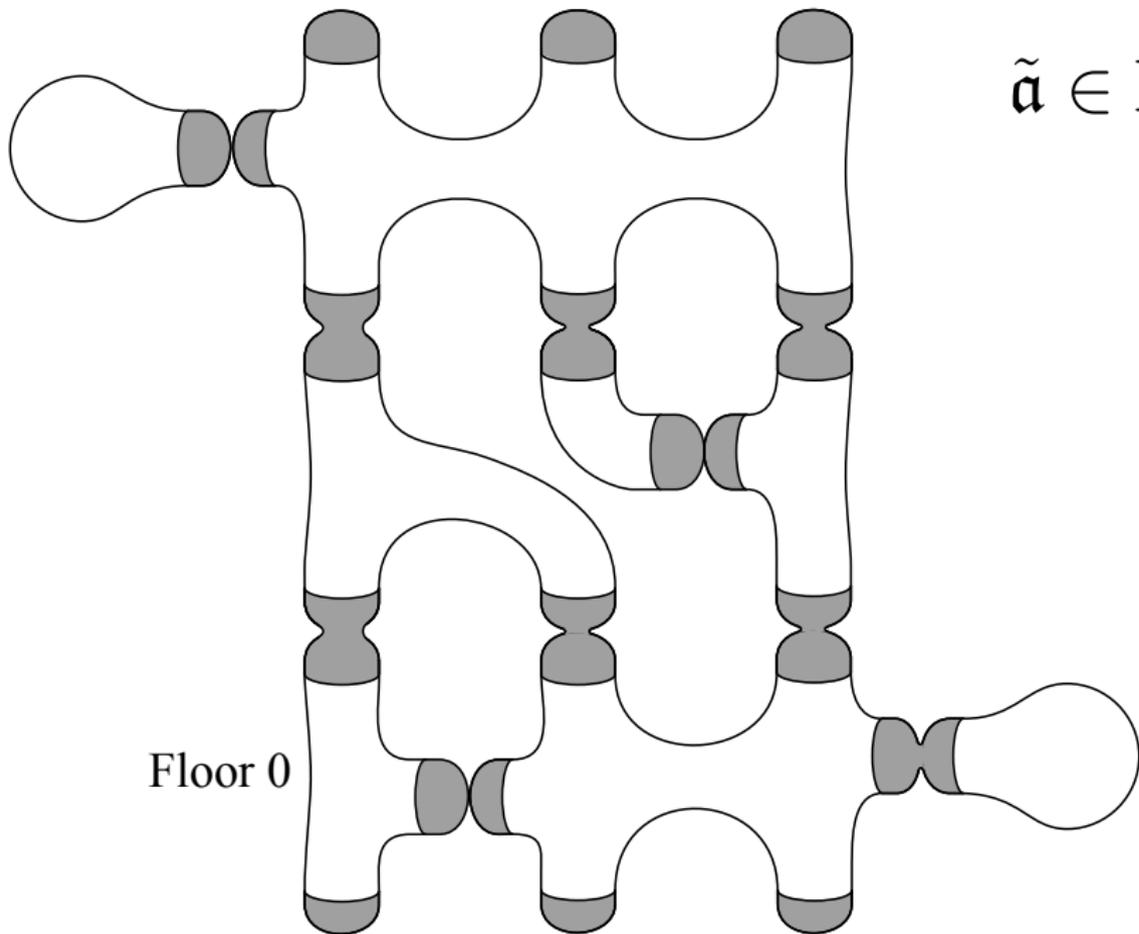
Floor 0



$$\sigma_{\tilde{\mathbf{a}}} = (\sigma_0^{\tilde{\mathbf{a}}})$$

$$\tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}}$$

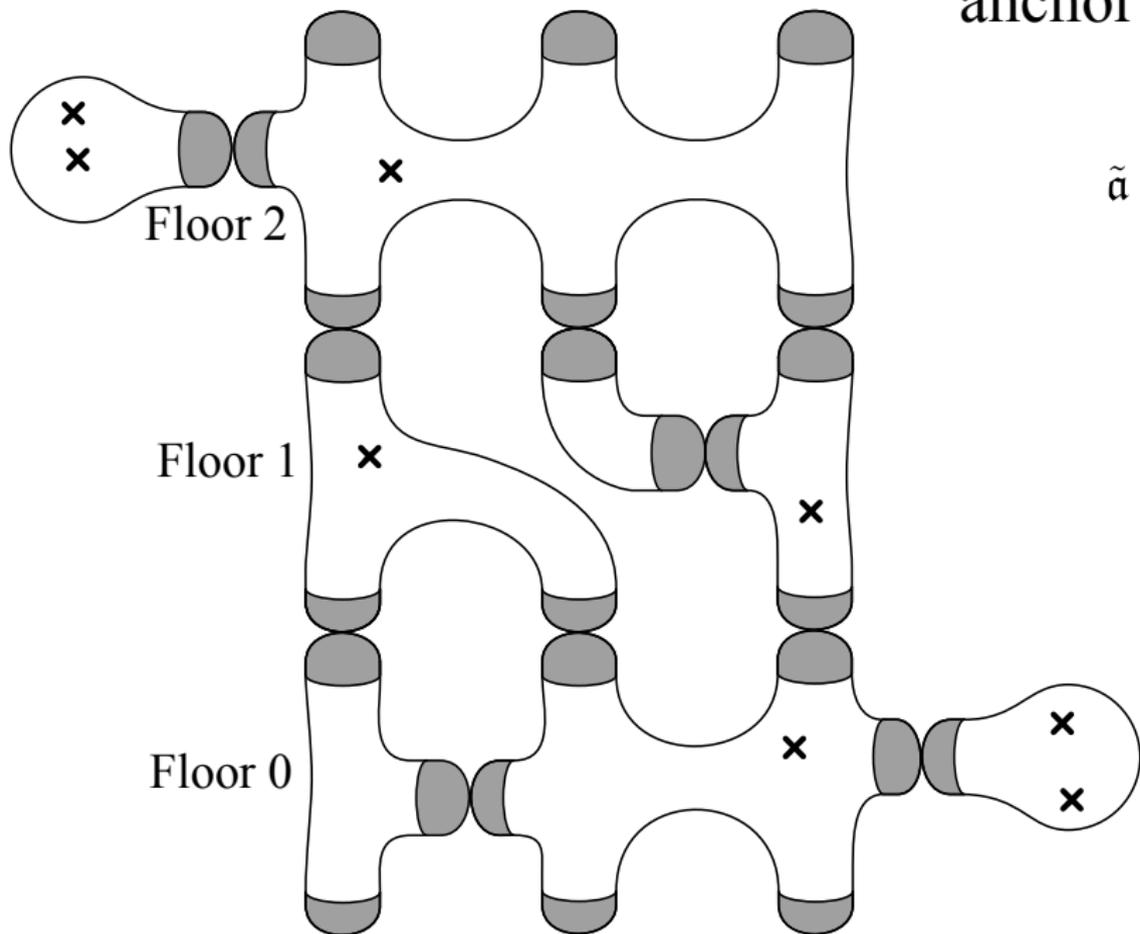
Floor 0



$$\sigma = (\sigma_0, b_1, \sigma_1, b_2, \sigma_2)$$

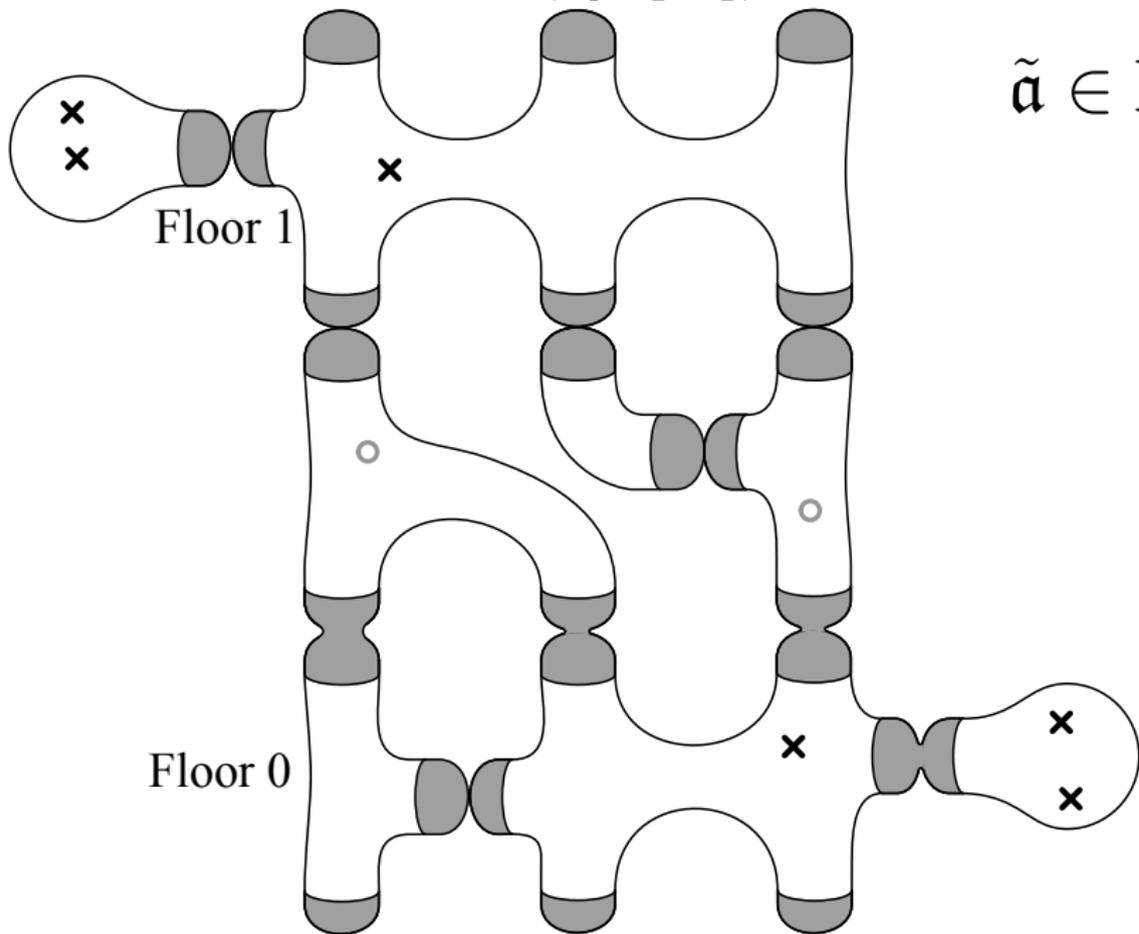
anchor points

$$\tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}}$$



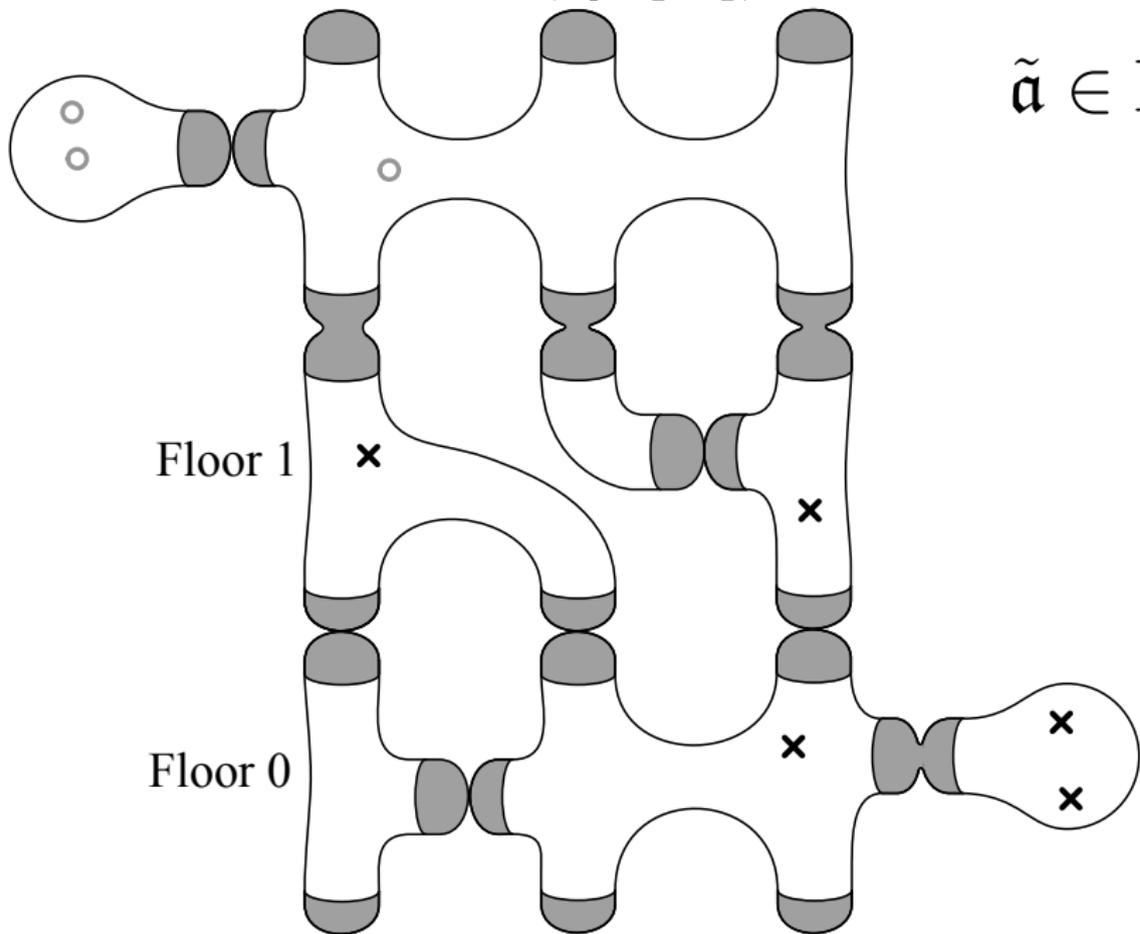
$$\sigma_{\tilde{a}} = (\sigma_0^{\tilde{a}}, b_1^{\tilde{a}}, \sigma_1^{\tilde{a}})$$

$$\tilde{a} \in \mathbb{B}_{\text{ad}}$$



$$\sigma_{\tilde{a}} = (\sigma_0^{\tilde{a}}, b_1^{\tilde{a}}, \sigma_1^{\tilde{a}})$$

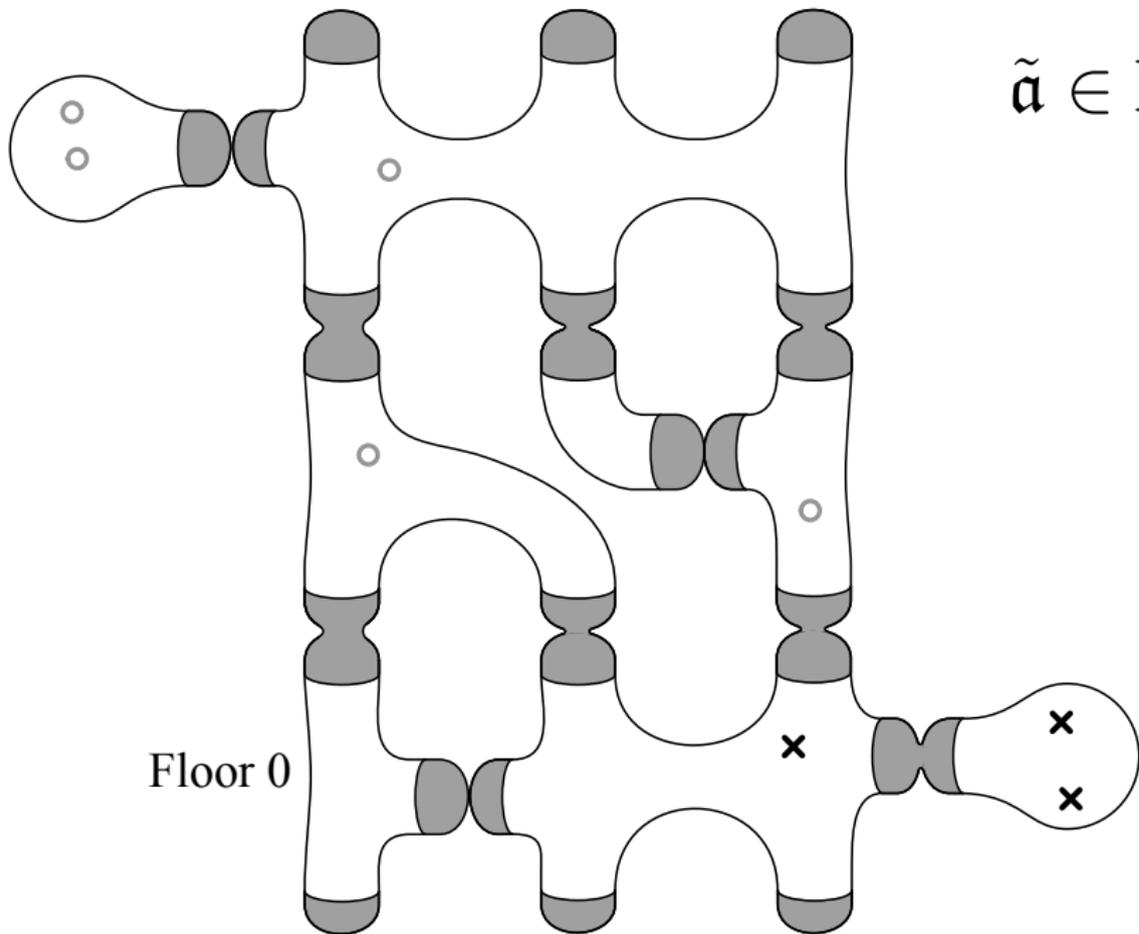
$$\tilde{a} \in \mathbb{B}_{\text{ad}}$$



$$\sigma_{\tilde{\mathbf{a}}} = (\sigma_0^{\tilde{\mathbf{a}}})$$

$$\tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}}$$

Floor 0



Recall:

$$\alpha = (\alpha_0, \hat{b}_1, \alpha_1, \hat{b}_2, \dots, \hat{b}_k, \alpha_k)$$

$$\alpha_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, [\tilde{u}_i], \Gamma_i^+)$$

$$\sigma_i = (\Gamma_i^-, S_i, j_i, M_i, D_i, \Gamma_i^+)$$

Then up to rearrangement:

$$\alpha = ((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, ([\tilde{u}_i])_{i=0}^k)$$

$$\downarrow + (\mathcal{J}_i)_{i=0}^k$$

$$((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, \underline{([\tilde{u}_i])_{i=0}^k}, (\mathcal{J}_i)_{i=0}^k)$$

Then up to rearrangement:

$$\alpha = \left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, ([\tilde{u}_i])_{i=0}^k \right)$$

$$\downarrow + (\mathcal{J}_i)_{i=0}^k$$

$$\left((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, \underline{(\tilde{u}_i)_{i=0}^k}, (\mathcal{J}_i)_{i=0}^k \right)$$

$$+ (\mathbf{D}_i)_{i=0}^k + (\mathbf{D}_i^+)_{i=0}^{k-1} + (\mathbf{D}_i^-)_{i=1}^k$$

$$+ (r_i)_{i=1}^k \in [0, 1)^k \quad \text{and} \quad \mathbf{a} \in \mathbb{B}_{\mathcal{D}}$$

conditioned on being in $\mathbb{B}_{\mathcal{D}} \times \mathcal{O}$

$$\hat{b}_i|_z \rightsquigarrow [\hat{x}, \hat{y}]_{(z, b_i(z))} \quad \text{with } z \in \Gamma_i^+$$

$$(z, \tilde{u}_i, \tilde{u}_{i+1}, r_i) \rightsquigarrow |a_{(z, b_i(z))}| \quad \text{via}$$

$$T_{\mathbf{F}(z, b_i(z))} \cdot \varphi(|a_{(z, b_i(z))}|) = \varphi(r_i) \ominus c_{\tilde{u}_i}^z \oplus c_{\tilde{u}_{i+1}}^{b_i(z)}$$

$$((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, (\tilde{u}_i)_{i=0}^k, (\mathcal{J}_i)_{i=0}^k)$$

$$+ (\mathbf{D}_i)_{i=0}^k + (\mathbf{D}_i^+)_{i=0}^{k-1} + (\mathbf{D}_i^-)_{i=1}^k$$

$$+ (r_i)_{i=1}^k \in [0, 1)^k \quad \text{and} \quad \mathbf{a} \in \mathbb{B}_{\mathcal{D}}$$

conditioned on being in $\mathbb{B}_{\mathcal{D}} \times \mathcal{O}$

$$\hat{b}_i|_z \rightsquigarrow [\hat{x}, \hat{y}]_{(z, b_i(z))} \quad \text{with } z \in \Gamma_i^+$$

$$(z, \tilde{u}_i, \tilde{u}_{i+1}, r_i) \rightsquigarrow |a_{(z, b_i(z))}| \quad \text{via}$$

$$T_{\mathbf{F}(z, b_i(z))} \cdot \varphi(|a_{(z, b_i(z))}|) = \varphi(r_i) - c^z(\tilde{u}_i) + c^{b_i(z)}(\tilde{u}_{i+1})$$

$$\rightsquigarrow |a| \cdot [\hat{x}, \hat{y}] = a$$

$$((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, (\tilde{u}_i)_{i=0}^k, (\mathcal{J}_i)_{i=0}^k, \tilde{\mathbf{a}} \in \mathbb{B}_{\text{ad}})$$

$$\rightsquigarrow |a| [\hat{x}, \hat{y}] = a$$

$$((\sigma_i)_{i=0}^k, (\hat{b}_i)_{i=1}^k, (\tilde{u}_i)_{i=0}^k, (\mathcal{J}_i)_{i=0}^k, \underline{\tilde{\mathbf{a}}} \in \mathbb{B}_{\text{ad}})$$



$$(\underline{(\sigma_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathbf{a}},e})_{e=1}^\ell, (\mathcal{J}_{\tilde{\mathbf{a}},e})_{e=0}^\ell}, (\tilde{u}_i)_{i=0}^k, (\mathcal{J}_{\tilde{\mathbf{a}},i}^{\text{vir}})_{i=0}^k, \tilde{\mathbf{a}})$$



for $i_e \leq i < i_{e+1}$

$$\tilde{u}_i^* = \begin{cases} \tilde{u}_i & \text{if } i = i_e \\ (\varphi(r_{i_e+1}) + \cdots + \varphi(r_i)) * \tilde{u}_i & \text{otherwise} \end{cases}$$

$$\tilde{w}_e = \oplus_{\tilde{\mathbf{a}}_e} (\tilde{u}_{i_e}^*, \dots, \tilde{u}_{i_{e+1}-1}^*)$$

$$((\sigma_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathbf{a}},e})_{e=1}^\ell, \underline{(\tilde{w}_e)_{e=0}^\ell}, (\mathcal{J}_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\mathcal{J}_{\tilde{\mathbf{a}},i}^{\text{vir}})_{i=0}^k, \tilde{\mathbf{a}})$$

$$((\sigma_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathbf{a}},e})_{e=1}^\ell, (\mathcal{J}_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\tilde{u}_i)_{i=0}^k, (\mathcal{J}_{\tilde{\mathbf{a}},i}^{\text{vir}})_{i=0}^k, \tilde{\mathbf{a}})$$

for $i_e \leq i < i_{e+1}$

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$$\tilde{w}_e = \oplus_{\tilde{\mathbf{a}}_e} (\tilde{u}_{i_e}^*, \dots, \tilde{u}_{i_{e+1}-1}^*)$$

$$((\sigma_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\hat{b}_{\tilde{\mathbf{a}},e})_{e=1}^\ell, (\tilde{w}_e)_{e=0}^\ell, (\mathcal{J}_{\tilde{\mathbf{a}},e})_{e=0}^\ell, (\mathcal{J}_{\tilde{\mathbf{a}},i}^{\text{vir}})_{i=0}^k, \tilde{\mathbf{a}})$$

$$\in Z_{\sigma, \mathcal{J}, \varphi}^{3, \delta_0}(\mathbb{R} \times \mathbb{R}^N, \mathbf{F})$$

Workhorse Imprinting Theorem:

This defines an imprinting

$$\bar{\oplus} : \mathbb{B}_D \times \mathcal{O} \rightarrow Z_{\sigma, \mathcal{J}, \varphi}^3(\mathbb{R} \times \mathbb{R}^N, \bar{\mathbf{F}})$$

Moreover this functorially extends to an imprinting for

$$Z_{\sigma, \mathcal{J}, \varphi}^3(\mathbb{R} \times Q, \bar{\mathbf{F}})$$

Building -- Road Map

✓ (stable map)

✓ (domain curve with floor str.)

✓
(domain curve)
(set of nodal gl. param.)
(subset of target gl. param.)

+

✓
(small disk str.)
(anchor points)

✓ (domain of imprinting)

✓
gluing parameters:
target \longrightarrow domain

✓ (target of imprinting)

✓ imprinting map naturally defined

Transversal Constraints:

Consider a map in $Z_{\sigma, \mathcal{X}, \varphi}^3(\mathbb{R} \times Q, \overline{\mathbf{F}})$ fix a G invariant finite set $\Xi = \Xi_0 \cup \dots \cup \Xi_k$ disjoint from usual interesting sets. For $z \in \Xi_i$ let $[z]$ denote its G orbit. There are two types of constraints:

- \mathbb{R} invariant:

Fix co-dimension 2 submanifold $H_{[z]} \subset Q$

$$\tilde{H}_{[z]} := \mathbb{R} \times H_{[z]}$$

- non \mathbb{R} invariant:

Fix co-dimension 1 submanifold $H_{[z]} \subset Q$

$$\tilde{H}_{[z]} := \{\bar{a}_{[z]}\} \times H_{[z]}$$

Transversal Constraints:

This yields an assignment: $\mathcal{H} : z \mapsto \tilde{H}_{[z]}$

Then define the subset $Z_{\sigma, \mathcal{J}, \mathcal{H}, \varphi}^3(\mathbb{R} \times Q, \overline{\mathbf{F}})$ of

$Z_{\sigma, \mathcal{J}, \varphi}^3(\mathbb{R} \times Q, \overline{\mathbf{F}})$ as those \tilde{w} for which

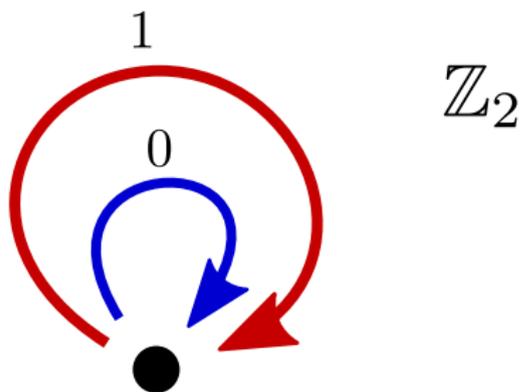
- $(-\text{av}_{\mathcal{J}_i}(\tilde{w})) * \tilde{w}(z) \in \tilde{H}_{[z]}$
- The above shifted map transversally intersects $\tilde{H}_{[z]}$

Definition -- Groupoidal Category

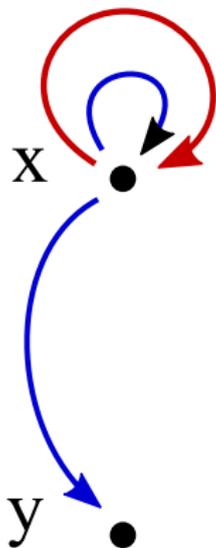
A groupoidal category is a category \mathcal{C} is a category with the following properties.

1. Every morphism is an isomorphism (i.e. has an inverse).
2. Between any two objects there are only finitely many morphisms.
3. The orbit space $|\mathcal{C}|$, (collection of isomorphism classes) is a set

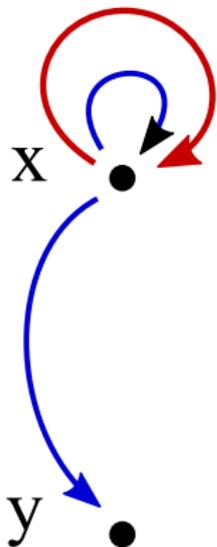
Toy Groupoidal Categories



Toy Groupoidal Categories



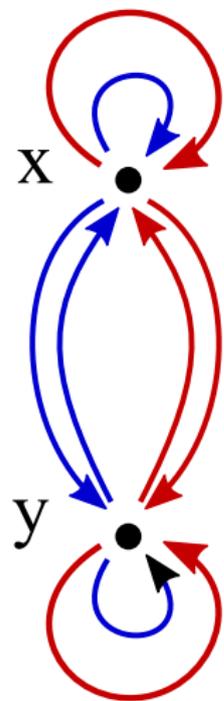
Toy Groupoidal Categories



Without adding objects,

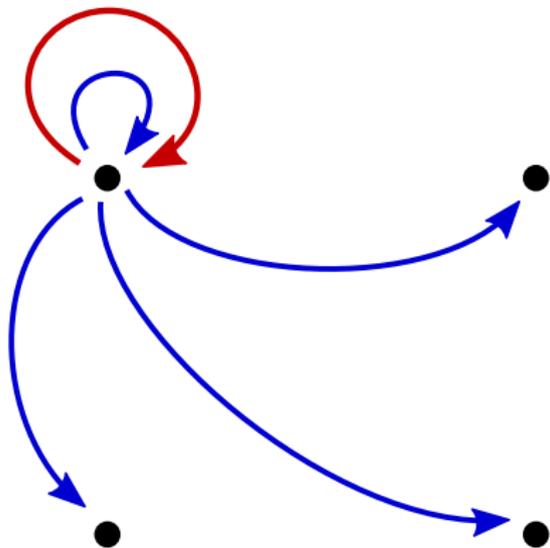
- 1) add the fewest morphisms to make this a groupoidal category
- 2) and without increasing the isotropy at x , add the most morphisms while keeping it a groupoidal category

Toy Groupoidal Categories



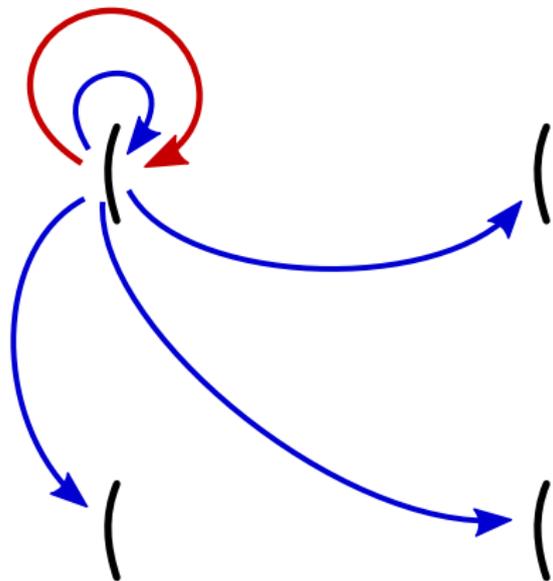
Answer

Toy Groupoidal Categories



Same questions:

Toy Groupoidal Categories



Same questions:

Representative Model

\mathcal{C}



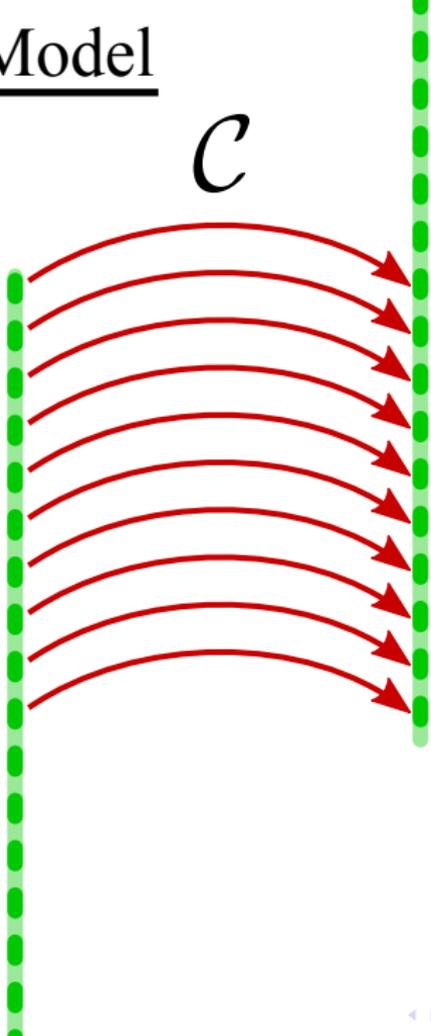
Representative Model

\mathcal{C}



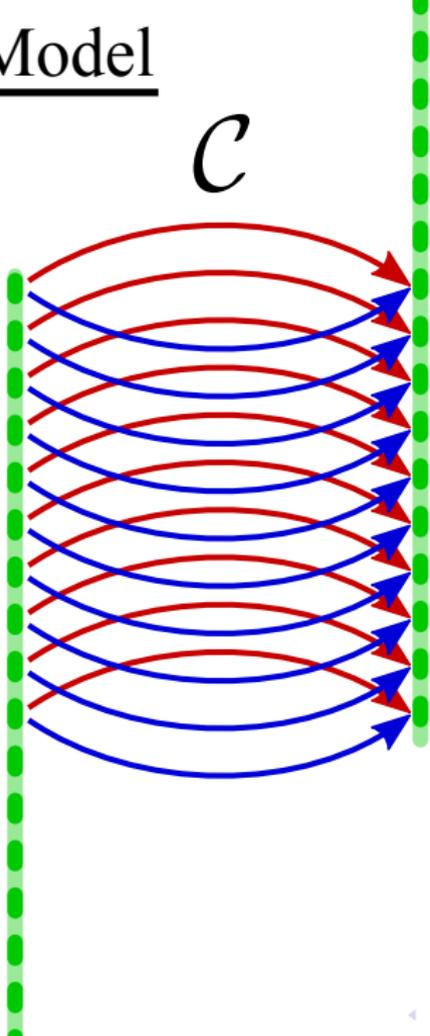
Representative Model

C



Representative Model

C



A Polyfold Goal:

We aim to equip the category of stable maps \mathcal{S} with a polyfold structure. This amounts to equipping it with

1. the structure of a GCT
2. a local uniformizer construction
3. M-polyfold structures on transition sets
4. with the compatibility conditions:
 - (a) the source and target maps are local sc-diffeomorphisms
 - (b) the other structure maps (unit, inversion, and multiplication) are sc-smooth

Definition -- GCT

A GCT is a pair $(\mathcal{C}, \mathcal{T})$ where \mathcal{C} is a groupoidal category and \mathcal{T} is a metrizable topology on the orbit space $|\mathcal{C}|$.

A Polyfold Goal:

We aim to equip the category of stable maps \mathcal{S} with a polyfold structure. This amounts to equipping it with

1. the structure of a GCT
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 - (a) the source and target maps are local sc-diffeomorphisms
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Definition -- Uniformizer

Given groupoidal category \mathcal{C} , a uniformizer at $c \in \text{Ob}(\mathcal{C})$ with automorphism group G , is a functor $\Psi : G \times O \rightarrow \mathcal{C}$ with the following properties.

1. O is an M-polyfold
2. G acts on O via sc-diffeomorphism
3. $G \times O$ is assoc. translation groupoid
4. there exists $\bar{o} \in O$ s.t. $\Psi(\bar{o}) = c$
5. Ψ is injective on objects
6. Ψ is full and faithful

Definition -- Local Uniformizer

Given a GCT $(\mathcal{C}, \mathcal{T})$, a uniformizer Ψ at $c \in \text{Ob}(\mathcal{C})$ is a *local* uniformizer provided $|\Psi| : |G \rtimes O| \rightarrow |\mathcal{C}|$ is a homeomorphism onto an open neighborhood of $|c|$.

Definition -- (local) Uniformizer Construction

A (local) uniformizer construction is a functor $F : \mathcal{C} \rightarrow \text{SET}$ which associates to an object c a set of (local) uniformizers.

If, for each object c , the set $F(c)$ contains only tame (local) uniformizers, then we shall call F a *tame (local) uniformizer construction*.

A Polyfold Goal:

We aim to equip the category of stable maps \mathcal{S} with a polyfold structure. This amounts to equipping it with

1. the structure of a GCT
2. a local uniformizer construction
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 - (a) the source and target maps are local sc-diffeomorphisms
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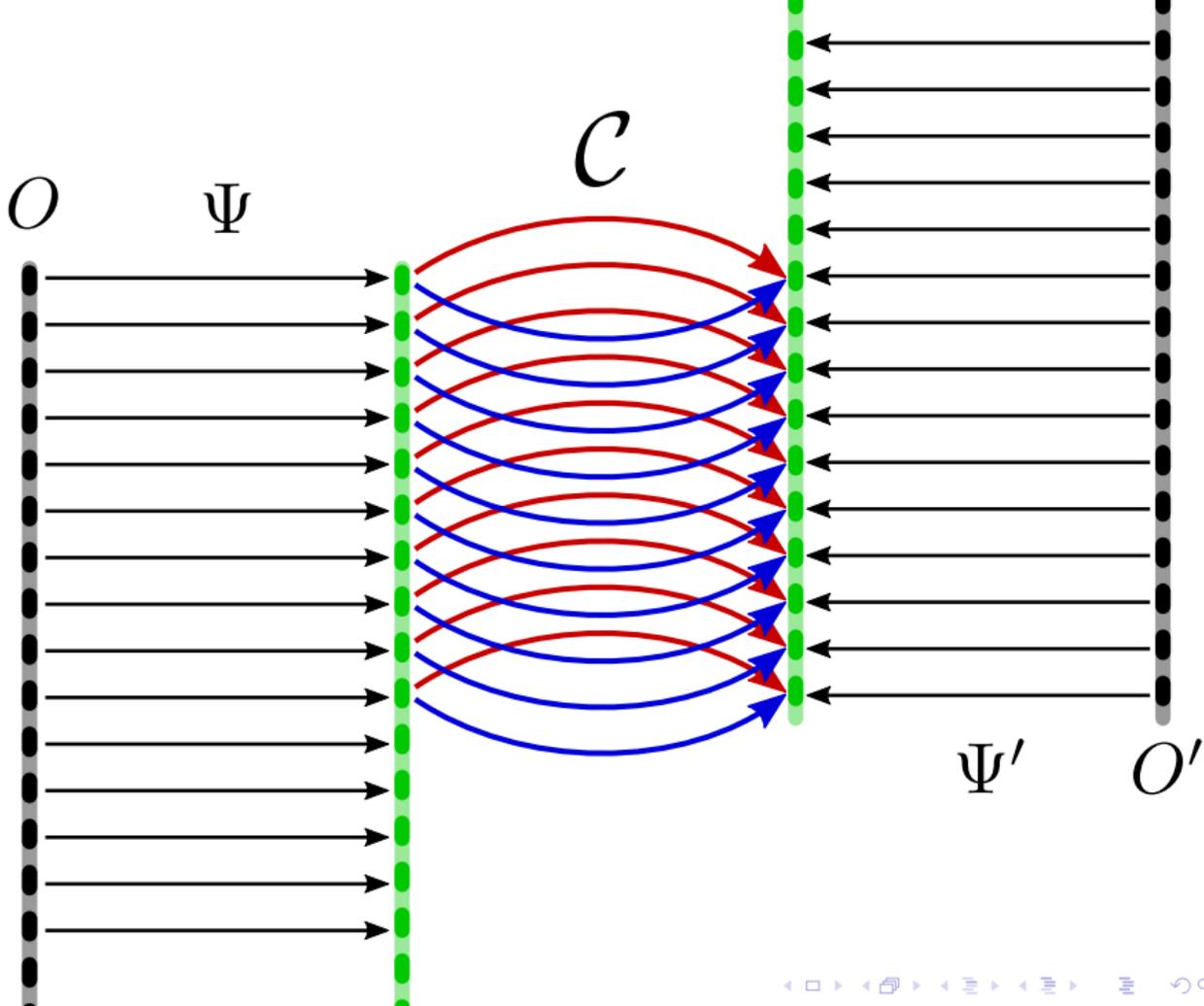
Definition -- Transition Set

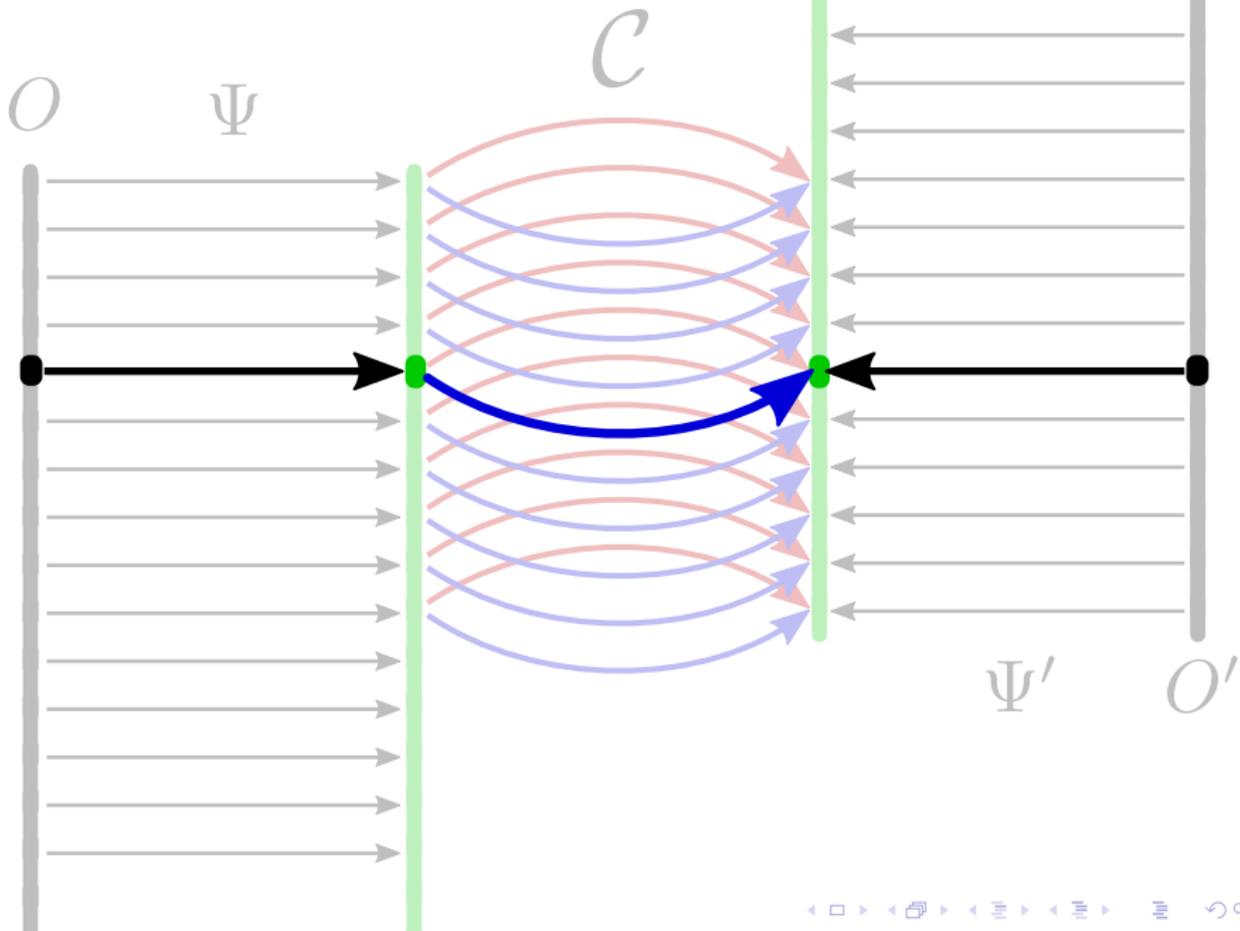
Fix a groupoidal category \mathcal{C} and a local uniformizer construction $F : \mathcal{C} \rightarrow \text{SET}$, $\alpha, \alpha' \in \text{Ob}(\mathcal{C})$, and local uniformizers $\Psi \in F(\alpha)$ and $\Psi' \in F(\alpha')$, so that

$$G \times O \xrightarrow{\Psi} \mathcal{C} \xleftarrow{\Psi'} G' \times O'$$

Define the transition set $\mathbf{M}(\Psi, \Psi')$ by

$$\mathbf{M}(\Psi, \Psi') = \left\{ (o, \Phi, o') : o \in O, o' \in O', \right. \\ \left. \Phi \in \text{Hom}(\Psi(o), \Psi'(o')) \right\}$$





"Transition Category"

Objects:

(Ψ, o) such that $\Psi : G \times O \rightarrow \mathcal{C}$
 $o \in O$

(Ψ', o') $\Psi' : G' \times O' \rightarrow \mathcal{C},$
 $o' \in O'$

Morphisms:

(o, Φ, o') such that $\Phi \in \text{Mor}(\mathcal{C})$
 $\Psi(o) \xrightarrow{\Phi} \Psi'(o')$

Definition -- Transition Set

$$\mathbf{M}(\Psi, \Psi') = \left\{ (o, \Phi, o') : o \in O, o' \in O', \right. \\ \left. \Phi \in \text{Hom}(\Psi(o), \Psi'(o')) \right\}$$

Recall that the transition set $\mathbf{M}(\Psi, \Psi')$ is equipped with the following structure maps.

1. **source map**
2. **target map**
3. **unit map** (identity)
4. **inversion map**
5. **multiplication map** (composition)

A Polyfold Goal:

We aim to equip the category of stable maps \mathcal{S} with a polyfold structure. This amounts to equipping it with

1. the structure of a GCT
2. a local uniformizer construction
3. M-polyfold structures on transition sets
4. with the compatibility conditions:
 - (a) the source and target maps are local sc-diffeomorphisms
 - (b) the other structure maps (unit, inversion, and multiplication) are sc-smooth

A Polyfold Goal:

Problem:

We don't have a topology on the orbit space.

Solution:

There exists a “work-around” construction which yields a natural topology.

Definition “work-around” construction consists of

1. (not local) uniformizer construction
2. transition germ construction

A Polyfold Goal:

Theorem (small miracle)

If a groupoidal category \mathcal{C} admits a “work-around” construction for which the associated natural topology is *metrizable*, then the “work-around” construction yields a polyfold structure on \mathcal{C} .

A Polyfold Goal:

Theorem

The groupoidal category of stable maps \mathcal{S} admits a “work-around” construction for which the associated natural topology is metrizable, and hence \mathcal{S} as a poyfold structure.

Transition Germ Construction

Let F be a uniformizer construction. A transition germ construction \mathcal{G} associates for given $\Psi \in F(c)$ and $\Psi' \in F(c')$ to $h = (o, \Phi, o') \in \mathbf{M}(\Psi, \Psi')$ a germ of map $\mathcal{G}_h : (\mathcal{O}, o) \rightarrow (\mathbf{M}(\Psi, \Psi'), h)$ with the following properties, where $\mathfrak{g}_h = t \circ \mathcal{G}_h$.

Transition Germ Construction

Diffeomorphism Property:

The germ $\mathfrak{g}_h : \mathcal{O}(O, o) \rightarrow \mathcal{O}(O', o')$ is a local sc-diffeomorphism and $s(\mathfrak{G}_h(q)) = q$ for q near o . If $\Psi = \Psi'$ and $h = (o, \Psi(g, o), g * o)$ then $\mathfrak{G}_h(q) = (q, \Psi(g, q), g * q)$ for q near o so that $f_h(q) = g * q$.

Transition Germ Construction

Stability Property:

$\mathfrak{G}_{\mathfrak{G}_h(q)}(p) = \mathfrak{G}_h(p)$ for q near $o = s(h)$ and p near q .

Transition Germ Construction

Identity Property:

$$\mathfrak{G}_{u(o)}(q) = u(q) \text{ for } q \text{ near } o.$$

Transition Germ Construction

Inversion Property:

$\mathfrak{G}_{\iota(h)}(\mathfrak{g}_h(q)) = \iota(\mathfrak{G}_h(q))$ for q near $o = s(h)$.

Here $\iota(p, \Phi, o') = (o', \Phi^{-1}, o)$.

Transition Germ Construction

Multiplication Property:

If $s(h') = t(h)$ then $\mathfrak{g}_{h'} \circ \mathfrak{g}_h(q) = \mathfrak{g}_{m(h',h)}(q)$ for q near $o = s(h)$, and $m(\mathfrak{G}_{h'}(\mathfrak{g}_h(q)), \mathfrak{G}_h(q)) = \mathfrak{G}_{m(h,h')}(q)$ for q near $o = s(h)$.

Transition Germ Construction

M-Hausdorff Property:

For different $h_1, h_2 \in \mathbf{M}(\Psi, \Psi')$ with $o = s(h_1) = s(h_2)$ the images under \mathcal{G}_{h_1} and \mathcal{G}_{h_2} of small neighborhoods are disjoint.